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ISOMORPHISMS OF FUNCTION MODULES, AND GENERALIZED APPROXIMATION IN MODULUS

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ABSTRACT. For a function algebra A we investigate relations between the following three topics: isomorphisms of singly generated A-modules, Morita equivalence bimodules, and "real harmonic functions" with respect to A. We also consider certain groups which are naturally associated with a uniform algebra A. We illustrate the notions considered with several examples.

1. INTRODUCTION

By a *uniform algebra* or *function algebra* on a compact Hausdorff space Ω, we shall mean a subalgebra A of $C(\Omega)$ (the continuous complex-valued functions on Ω) which contains constants and separates points. In most of this paper we are concerned with closed submodules of $C(\Omega)$ of the form Af, where f is a strictly positive and continuous function on Ω. In Part C we will allow f to be nonnegative.

Before we proceed any further with this introduction, we shall take a paragraph to explain why these modules are more general than they appear to be at first. We need a little notation. By a *concrete function A-module* we shall mean a closed linear subspace of $C(K)$, for a compact Hausdorff space K, which is closed under multiplication by $\pi(A)$, where $\pi : A \to C(K)$ is a unital homomorphism. By an *(abstract) function A-module* we shall mean a Banach A-module X which is isometrically A-module isomorphic to (in future we shall simply say "A-isometric to" for short) a concrete function module. Although we shall not particularly use this here, this class of modules was given several equivalent abstract characterizations in [9]; for example, it coincides with the class of Banach A-modules whose module action is contractive with respect to the injective tensor product. It is proved in [8] that any algebraically singly generated faithful[1] function A-module is A-isometric to one of the form Af described in the first paragraph, for some strictly positive continuous function f on *some* compact space Ω on which A is isometrically represented as a uniform algebra.

In this paper we investigate the relations between the following three topics: isometries and almost isometries between modules of the type discussed above, real "harmonic" functions on Ω with respect to A, and Morita equivalence bimodules over A.

In Part A we provide a necessary and sufficient condition on functions f_1, f_2 for the modules Af_1, Af_2 to be isometric (as Banach spaces). The result generalizes the classical description of isometries of uniform algebras. We then extend the result

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[1]A left module X is faithful if $aX = 0$ implies $a = 0$.

to almost isometric modules; here modules Af_1, Af_2 are called "almost isometric" if for any $\varepsilon > 0$ there is a surjective linear isomorphism $T : Af_1 \to Af_2$ such that $\|T\| \left\|T^{-1}\right\| < 1 + \varepsilon$.

One of the main results in Part A is a characterization of modules of the type Af which are almost isometric to A. If $\Omega = \partial A$, the Shilov boundary of A, then we will see that "$Af \cong A$ almost isometrically" is equivalent to f being uniformly approximable by the moduli of invertible elements of A. That is, f is in the uniform closure $\bar{Q}$ of Q, where $Q = \{|a| : a \in A^{-1}\}$. Indeed, up to A-isometric isomorphism, for any Ω on which A is a function algebra, the Banach A-modules which are almost A-isometric to A are exactly the submodules $Af \subset C(\Omega)$, for some strictly positive $f \in \bar{Q}$. Thus there is a correspondence (which is 1-1 modulo the group Q) between (equivalence classes up to A-isometric isomorphism of) Banach A-modules which are almost A-isometric to A, and the set $\{f : 0 < f \in \bar{Q}\}$. In turn, this last set is, via the log function, in a one-to-one correspondence with the set $H_A(\Omega) = \{\log f : 0 < f \in \bar{Q}\}$. Thus one may "label" any Banach A-module which is almost A-isometric to A, by a function in $H_A(\Omega)$. The next point is that the set $H_A(\Omega)$ is very well known to those familiar with the theory of uniform algebras. In particular, A is known as "logmodular" if $H_A(\Omega) = C_{\mathbb{R}}(\Omega)$. For any function algebra A, the class $H_A(\Omega)$ deserves to be called a "harmonic class" of functions with respect to A. In this paper, to be more specific, by a "harmonic class" we shall mean a class $\mathcal{B}(\Omega)$ of real continuous functions on Ω which have at least the following properties:

(i) If $f \in \mathcal{B}(\Omega)$, then $f_{|\partial A} \in \mathcal{B}(\partial A)$.
(ii) If $f \in \mathcal{B}(\Omega)$, then there exists a unique $\tilde{f} \in \mathcal{B}(M_A)$ such that $\tilde{f}_{|\Omega} = f$. (M_A is the maximal ideal space of A.) Also, for any $g \in \mathcal{B}(M_A), g_{|\Omega} \in \mathcal{B}(\Omega)$.
(iii) Every $f \in \mathcal{B}(\Omega)$ achieves its maximum and minimum value on ∂A.
(iv) $f_1, f_2 \in \mathcal{B}(\Omega)$ implies $f_1 + f_2 \in \mathcal{B}(\Omega)$. (Indeed, most of the classes we study in this paper are additive groups.)
(v) If A consists of functions which are analytic on a region R in $\mathbb{C}^n$, and which separate points of R, then (via the obvious homeomorphic embedding $R \to M_A$) the functions in $\mathcal{B}(M_A)$ are genuinely harmonic on R.

This leads us conveniently into a description of Part B of the paper. Here is a natural idea to attempt to generalize the class $H_A(\Omega)$, or equivalently, the strictly positive functions which are uniform limits

$$f = \lim_n |k_n|, \text{ where } k_n \ , \ h_n \ \in A \text{ with } k_n h_n = 1, \tag{1}$$

on Ω. We will write $A^{(n)}$ for the space of n-tuples with entries in A. An element of $A^{(n)}$ will be called an A-tuple, and will often be regarded as a function $\Omega \to \mathbb{C}^n$. For two A-tuples $H = (h_i), K = (k_i)$ of the same "length" we define $H.K = \sum_i h_i k_i \in A$. Consider the set of strictly positive functions which are uniform limits

$$f(w) = \lim_n \|K_n(w)\|_2 \ , \text{ where } K_n \ , \ H_n \ \in A^{(m_n)} \text{ with } K_n.H_n = 1, \tag{2}$$

on Ω, where the $m_n \in \mathbb{N}$. This looks like a natural generalization of (1). However, one quickly sees that there is a hidden condition in (1) which is not in (2), which results in (2) not corresponding to a "harmonic class". Namely, in (1), because $h_n = k_n^{-1}$, we automatically have $|h_n| \to f^{-1}$ uniformly. We therefore define $\mathcal{M}_A(\Omega)$ (sometimes written $\mathcal{M}$ or $\mathcal{M}_A$ when there is no confusion) to be the set of strictly positive functions f on Ω which satisfy (2) and also

$$f(w)^{-1} = \lim_n \|H_n(w)\|_2 \quad (3)$$

uniformly on Ω. This might loosely be called a "tight convex approximation in modulus". One finds that now $\log \mathcal{M}_A(\Omega)$ is a "harmonic class", which contains $H_A(\Omega)$. This is shown in Part C. We must admit that we were not able to answer the question of whether $\log \mathcal{M}_A(\Omega) = H_A(\Omega)$ in general; this seems to be a very interesting open problem which may depend on deep complex geometry. This question would have interesting consequences whatever its answer turns out to be. In any case, even if the two classes do turn out to coincide, this would not adversely impact our paper beyond reducing it by a few pages in length!

We show that $f \in \mathcal{M}_A(\Omega)$ if and only if Af is a strong Morita equivalence A-A-bimodule, with "inverse bimodule" Af^{-1}. The notion of strong Morita equivalence for function and possibly nonselfadjoint operator algebras was defined and studied in [11], but for function algebras and singly generated A-modules X, this notion may be viewed as a generalization of the notion $X \cong A$. Indeed, X is a "rank one" strong Morita equivalence bimodule if and only if $X \cong A$ almost A-isometrically. We display certain groups that are naturally associated with a uniform algebra A, such as the Picard group. We make several observations on various kinds of approximations in modulus, and illustrate the notions considered with some interesting examples.

In Part C, we generalize still further. We now allow topologically singly generated function modules. These correspond to submodules $(Af)^{\bar{}}$, the closure taken in $C(\Omega)$, where f is now allowed to be a nonnegative continuous function on Ω. We define a larger harmonic class $\log \mathcal{R}(\Omega)$, which contains the harmonic classes mentioned earlier. Just as the $\mathcal{M}$ class corresponds to Morita equivalence, the $\mathcal{R}$ class corresponds to the more general notion of "rigged module". Rigged modules were intended to be a generalization of the notion of "Hilbert C^*-module" and were studied in [4, 11]; but in our (singly generated) situation these are the modules X for which the identity map $X \to X$ factors asymptotically, via contractive A-module maps, through the "free" A-modules $A^{(n)}$. This asymptotic factorization may be viewed as another generalization of the statement $X \cong A$ A-isometrically.

The reader will find in §8 the definition of "$f \in \mathcal{R}(\Omega)$"; we believe that it is an interesting function algebra/function theory problem to study and classify these functions further. For example, we show in §8 that for $A = A(\mathbb{D})$, the disk algebra, $f \in \mathcal{R}$ if and only if f is a continuous function on $\bar{\mathbb{D}}$ such that $f = |\phi|$ for some outer function $\phi \in H^\infty$. This class of functions f coincides with the continuous nonnegative functions on $\mathbb{T}$ with integrable logarithm. The topologically singly generated $A(\mathbb{D})$-rigged modules are exactly (up to A-isometric isomorphism) the modules of the form $(Af)^{\bar{}}$, where f is as in the last two lines above.

We admit that one of the purposes of Parts B and C was to begin to illuminate the function algebra case of the theory of Morita equivalence and rigged modules, which the first author and various coauthors have developed over the years (see [6] for a leisurely survey), and in particular to see the connections with the theory of function algebras, and some interesting questions about such algebras. Indeed, we believe that our paper does lead to some interesting open function theory problems. Conversely, what we do here may lead to progress in the noncommutative situation.

As was pointed out to us by T. J. Ransford, subharmonic functions in abstract settings have also been considered by other authors. In 1972 C. E. Rickart [30] investigated plurisubharmonic functions associated with "natural systems" of concrete Banach function algebras. Later T. W. Gamelin [18] and T. W. Gamelin and N. Sibony [19] studied the notion of subharmonicity defined for functions on a closed subset E of the maximal ideal space M_A of a uniform algebra A. They defined a class $\mathcal{SH}_A$ of subharmonic functions on E as uniform limits on E of functions of the form

$$\max\{-M, c_1 \log|f_1|, ..., c_m \log|f_m|\},$$

where M is real, $c_1, ..., c_m > 0$ and $f_1, ..., f_m \in A$. Gamelin and Sibony proved that the above definition directly extends the concept of the "classical" plurisubharmonic functions: if a compact subset K of $\mathbb{C}^n$ is a decreasing limit of open unions of domains of holomorphy, if ∂K is smooth, and if $A = \mathrm{Hol}(K)$ is the closure in $C(K)$ of the algebra of functions analytic in a neighborhood of K, then $M_A = K$ and the class $\mathcal{SH}_A$ on K coincides with the class of continuous functions on K which are plurisubharmonic functions on $\mathrm{int}K$ (see [18], Theorems 6.10 and 6.12). By [18], Theorem 6.17, for a sufficiently regular subset K of $\mathbb{C}^n$, the restrictions of the functions from $\mathcal{SH}_A$ to $C_{\mathbb{R}}(\partial K)$ coincide with $C_{\mathbb{R}}(\partial K)$.

At least for algebras consisting of holomorphic functions on domains in $\mathbb{C}^n$, our class $\log\mathcal{M}$ can be shown to consist of pluriharmonic functions on the domain. Thus the functions we consider are certainly related to the setting of Gamelin and Sibony. Indeed, the last observation suggests the conjecture that if f and $-f$ are subharmonic in their sense, then f is in our $\log\mathcal{M}$, but in fact this is false in general, as is clear from our example after Theorem 6.4. There are perhaps profitable connections here that should be explored in the future.

2. Notation and definitions

For a compact set Ω we denote by $C(\Omega)$ (resp. $C_{\mathbb{R}}(\Omega)$) the space of all complex-valued (resp. real-valued) continuous functions on Ω. For a function algebra A on Ω we will write M_A for the maximal ideal space of A, and ∂A for the Shilov boundary of A. Then $C(\partial A), C(M_A)$, and $C(\Omega)$ may be regarded, respectively, as the minimal, maximal commutative, and generic commutative C^*-algebra generated by A. A may be viewed as a closed subalgebra of continuous functions on any of these three compact spaces. For a set of functions $\mathcal{E}$ we will write $\mathcal{E}_+$ for the nonnegative functions in $\mathcal{E}$, and $\mathcal{E}^+$ for the strictly positive functions in $\mathcal{E}$. We will refer very often to the following important subsets of $C(\Omega)$ which may be associated with A:

- A^{-1} is the set of invertible elements of A,
- $P = \{f \in C(\Omega) : f = |g|, g \in A\}$,
- $Q = \{f \in C(\Omega) : f = |g|, \text{ where } g \in A^{-1}\}$,
- $\mathcal{F} = \{f = \sum_{i=1}^n |g_i|^2 \in C(\Omega) : n \in \mathbb{N}, g_1, \cdots g_n \in A\}$,
- $\mathcal{G} = \bar{\mathcal{F}}$, the uniform closure of $\mathcal{F}$ in $C(\Omega)$,
- $GH_A(\Omega)$ is the closure of $\mathrm{Re}A$ in $C_{\mathbb{R}}(\Omega)$,
- $H_A(\Omega)$ is the closure of $\{\log|a| : a \in A^{-1}\}$ in $C_{\mathbb{R}}(\Omega)$,
- $\mathcal{M}_A(\Omega)$ is the set of $f \in C(\Omega)^+$ satisfying conditions (2) and (3) from §1. The limits here are uniform limits.

If we wish to specify the dependence on Ω, we will write, for example, $Q(\Omega)$ or $Q_A(\Omega)$. We will see in §6 that $\mathcal{M}_A(\Omega) \neq \{f : f, f^{-1} \in \mathcal{G}_A(\Omega)\}$ in general. It is well known that $GH_A(\Omega) \subset H_A(\Omega)$, and they are both harmonic classes in the sense of the introduction (a proof is also contained in Part C). We recall that A is called a "Dirichlet algebra" (resp. "logmodular algebra") if $GH_A(\partial A) = C_{\mathbb{R}}(\partial A)$ (resp. $H_A(\partial A) = C_{\mathbb{R}}(\partial A)$). The disk algebra $A(\mathbb{D})$ is Dirichlet (and consequently logmodular). Indeed, this is exactly the same as saying that the ordinary Dirichlet problem (of harmonic extension from the boundary) can be solved on the circle $\mathbb{T} = \partial A$. This, of course, was Gleason's original reason for the name "Dirichlet algebra" (see [20]).

An argument using elementary one-variable real analysis shows that the bijection $C(\Omega)^+ \to C_{\mathbb{R}}(\Omega)$ given by the log function maps the set $\bar{Q}^+$ onto the set $H_A(\Omega)$, using the definition of the latter set from a paragraph back. This observation shows that the definition of $H_A(\Omega)$ given in the introduction coincides with the last definition, and also shows that A is logmodular if and only if $C(\Omega)^+ = \bar{Q}^+$.

We write $\hat{}\colon A \to C(M_A)$ for the Gelfand transform, and we often use the same symbol to denote an element of an algebra and the corresponding Gelfand transform. There is a surprising equivalence relation on M_A: $\phi, \psi \in M_A$ are equivalent if and only if $\|\phi - \psi\| < 2$. The distinct equivalence classes are called the (Gleason) parts [20]. They are $\sigma-$compact subsets of M_A. We refer the reader to [34] or [17] as general references on function algebras.

If X_1 and X_2 are Banach A-modules, then we write $X_1 \cong X_2$ A-isometrically (resp. almost A-isometrically), if they are isometrically A-isomorphic (resp. almost isometrically A-isomorphic). As we said earlier, the latter term means that for any $\epsilon > 0$, there exists an A-module isomorphism $T : X_1 \to X_2$ with $\|T\|\|T^{-1}\| < 1+\epsilon$. If, in the above, we replace the requirement that T be an A-module map by its being linear, then we simply say that $X_1 \cong X_2$ almost isometrically.

Whenever we use the words "singly generated", it will be assumed, unless otherwise qualified, to have the topological connotation. Thus a "singly generated" Banach module X over A has an element x such that the closure of Ax is X. An *algebraically* singly generated module X has $Ax = X$ for some $x \in X$.

We will write $A^{(n)}$ for the space of n-tuples with entries in A. An element of $A^{(n)}$ will be called an A-tuple, and will often be regarded as a function $\Omega \to \mathbb{C}^n$.

For a Banach space X, we let X_1 be the unit ball of X, and we denote by $ext(X_1)$ the set of extreme points of the unit ball of X. If A is a subspace of $C(\Omega)$ and if x is a point in Ω, we use the same symbol x to denote the corresponding functional on A—namely, evaluation at x. There are several definitions of the Choquet boundary $Ch(A)$ of a linear subspace A of $C(\Omega)$; here we adopt the following definition:

$$Ch(A) = \{x \in \Omega : x \in ext(A_1^*)\}. \tag{4}$$

The Shilov boundary ∂A is the closure of $Ch(A)$. As it may be necessary to distinguish between functions from A and their restrictions to ∂A, we will denote by $A_{|\partial A}$ the set of these restrictions.

In the latter part of the paper, we will be working with operator spaces. However, usually, issues of "complete boundedness" do not arise. This is because for a linear operator T mapping into a subspace of a commutative C^*-algebra we have $\|T\| = \|T\|_{cb}$.

Part A

3. Module isomorphisms

It is well known [28] that two uniform algebras are linearly isometric, that is, isometric as Banach spaces, if and only if they are isomorphic as algebras. In this section we show that similar results hold for function modules of the form Af. To explain the main idea, let A be a uniform algebra on Ω, and let f_1, f_2 be strictly positive continuous functions on Ω. Suppose that there is an invertible function $g \in A$ such that $|g| = \frac{f_1}{f_2}$. Then

$$Af_1 \ni af_1 \longmapsto agf_2 \in Af_2$$

is an A-module isometry between Af_1 and Af_2. Suppose now that we also have a homeomorphism φ of Ω onto itself such that $\{a \circ \varphi : a \in A\} = A$, and $\frac{f_1 \circ \varphi}{f_2} = |h|$, for some invertible element h of A. Then

$$Af_1 \ni af_1 \longmapsto (a \circ \varphi) h f_2 \in Af_2 \tag{5}$$

is a linear isometry between Af_1 and Af_2, but this time it is not an A-module isometry unless φ is the identity map. We shall show that any linear isometry between modules Af_1 and Af_2 is essentially of the form (5).

We first need to introduce some technical results.

Lemma 3.1. *Suppose that A is a linear subspace of $C(\Omega)$ separating the points of Ω. Then:*

- **(i)** *the original topology τ of Ω is identical with the weak* topology σ^* on Ω considered as a subset of the unit ball of the dual space A^*;*
- **(ii)** *if F is an extreme point of the unit ball of the dual space A^*, then there are a point x in $Ch(A)$ and a scalar α of absolute value one such that $F = \alpha x$;*
- **(iii)** *for any $a \in A$ we have*

$$\|a\| = \sup\{|a(x)| : x \in Ch(A)\}. \tag{6}$$

Proof. **(i)** It is clear that the identity map $I : (\Omega, \tau) \to (\Omega, \sigma^*)$ is continuous. Since (Ω, τ) is compact, it follows that the topologies are identical.

(ii) Assume that F is an extreme point of the unit ball of the dual space A^*. By the Krein-Milman Theorem, the set of norm-one extensions of F to $C(\Omega)$ has an extreme point μ. It is easy to check that μ is also an extreme point of $C(\Omega)_1^*$. The argument is concluded by appealing to the very well-known fact that any extreme point of the last space is of the form αx with $x \in \Omega, |\alpha| = 1$.

(iii) Let a be a norm-one element of A. By the Krein-Milman Theorem there is an extreme point F of the unit ball of A^* such that $F(a) = 1$. Thus (6) follows from the previous part. □

Lemma 3.2. *Suppose that A is a uniform algebra on Ω, that x_0 is in the Choquet boundary of A, and that f and p are strictly positive continuous functions on Ω. Then there is an $a \in A$ such that $(af)(x_0) = p(x_0)$ and $|af| \le p$.*

Moreover, only points from the Choquet boundary have this property.

Proof. Follows from [17], II.12. □

Lemma 3.3. *Suppose that A is a uniform algebra on Ω and that f is a strictly positive continuous function on Ω. Then $Ch(A) \subset Ch(Af) \subset \Omega$.*

Proof. To prove the first inclusion, assume that $x \in Ch\,(A)$ and $x = \frac{1}{2}F_1 + \frac{1}{2}F_2$, where F_1, F_2 are norm-one functionals on Af. Let μ_1, μ_2 be norm-one extensions of F_1, F_2 to functionals on $C\,(\Omega)$. By Lemma 3.2, there is a net a_γ in A such that $\|a_\gamma\| = a_\gamma\,(x) = 1$ and $a_\gamma \to 0$ uniformly on compact subsets of $\Omega \setminus \{x\}$. We have

$$f\,(x) = (a_\gamma f)\,(x) = \frac{1}{2}\int_\Omega a_\gamma f d\mu_1 + \frac{1}{2}\int_\Omega a_\gamma f d\mu_2 \to \frac{f\,(x)}{2}\,(\mu_1 + \mu_2)\,(\{x\}),$$

and hence $\mu_1\,(\{x\}) = 1 = \mu_2\,(\{x\})$. Since $\|\mu_i\| \leq 1$, we get $\mu_1 = x = \mu_2$. ☐

Notice that, in general, $Ch(A)$ may be a proper subset of $Ch(Af)$. If, for example, A is equal to the disk algebra, if Ω is the closed unit disk $\bar{\mathbb{D}}$, and if $f\,(z) = 2 - |z|$, then $Ch\,(A) = \partial A = \partial\mathbb{D}$, while $0 \in Ch(Af)$.

Theorem 3.4. *Assume that A and B are uniform algebras on compact sets Ω_1, Ω_2 respectively, and that f_1, f_2 are strictly positive continuous functions on Ω_1, Ω_2 respectively. Suppose that there is a surjective linear isometry $T : Af_1 \to Bf_2$. Then there are an invertible element h of B and a homeomorphism φ of Ω_2 onto a subset $\varphi\,(\Omega_2)$ of M_A, such that*

(i) *the map $a \mapsto a \circ \varphi$ is an isometric isomorphism of A onto B;*
(ii) *$\varphi\,(\partial B) = \partial A$;*
(iii) *$\frac{f_1 \circ \varphi}{f_2}{}_{|\partial(Bf_2)} = |h|_{|\partial(Bf_2)}$, and*

$$T\,(af_1) = (a \circ \varphi)hf_2 \quad \text{on } \Omega_2\,, \text{ for } \; a \in A. \tag{7}$$

Moreover, if $A = B$, then T is an A-module isometry if and only if φ is equal to the identity map.

Before we prove this theorem, we give some consequences.

Corollary 3.5. *Assume that A is a uniform algebra on a compact set Ω, and that f is a strictly positive continuous function on Ω. Then A and Af are linearly isometric if and only if they are A-isometric.*

Proof. Assume that $T : A \to Af$ is a linear isometry. Set $f_1 = 1, f_2 = f$ in the last theorem, and let h be as in that theorem. We can define a module map $S : A \to Af$ by

$$S\,(a)\,(x) = ahf\,(x)\,, \qquad \text{for } a \in A, \quad x \in \Omega.$$

That S is an isometry follows from the last theorem and Lemma 3.1 (iii). ☐

In general it is not true that modules Af_1 and Af_2 are linearly isometric if and only if they are A-module isometric. Even if $f_1, f_2 \in C(\partial A)^+$, this is not true. Assume, for example, that A is equal to the product of the disk algebra and the two-dimensional algebra $C\,(\{-1, 1\})$. Set $\Omega = \partial A = \mathbb{T} \times \{-1, 1\}$, and define a map $\varphi : \Omega \to \Omega$ by $\varphi\,(z, j) = (z, -j)$, let $f \in C\,(\mathbb{T})^+ \setminus Q(A(\mathbb{D}))$, and put

$$f_1\,(z, j) = \begin{cases} 1 & \text{for } j = -1, \\ f\,(z) & \text{for } j = 1, \end{cases}$$

and $f_2\,(z, j) = f_1\,(z, -j)$. Define $T\,(af_1)\,(z, j) = a\,(z, -j)\,f_2\,(z, j)$. Then T is a linear isometry from Af_1 onto Af_2. Assume now that there is an A-module isometry $S : Af_1 \to Af_2$. By Theorem 3.4 there is an invertible element h of A such that

$$S(af_1) = ahf_2, \quad \text{for } a \in A.$$

Fix $(z, j) \in \mathbb{T} \times \{-1, 1\}$, and let a_n be a sequence of norm-one elements of A convergent to zero almost uniformly on $\mathbb{T} \times \{-1, 1\} \setminus \{(z, j)\}$. The norm of af_1 is convergent to $|f_1(z, j)|$, while the norm of ahf_2 is convergent to $|h(z, j) f_2(z, j)|$; hence

$$|h(z, j) f_2(z, j)| = |f_1(z, j)|, \text{ for } (z, j) \in \mathbb{T} \times \{-1, 1\}.$$

Hence $|f(z)| = |f_1(z, 1)| = |h(z, 1) f_2(z, 1)| = |h(z, 1)|$. However, $h(\cdot, 1)$ is an invertible element of the disk algebra, contrary to our assumption about f.

Corollary 3.6. *Assume that A is a uniform algebra on a compact set Ω, and that f is a strictly positive continuous function on Ω. Then the following conditions are equivalent:*

1. *$f \in Q$;*
2. *A, Af, and Af^{-1} are A-module isometric;*
3. *A, Af, and Af^{-1} are linearly isometric.*

Proof. The only thing that is still not clear here is that $(2 \Rightarrow 1)$. Note that if $T_1 : A \to Af$ is an A-isometric isomorphism, and if $T_1(1) = hf$, then $h \in A^{-1}$, $\|hf\| = 1$, and $|h|f \le 1$ on Ω. Since $\|a\| = \|ahf\|$ for all $a \in A$, it follows by a simple Choquet point argument that $|h|f = 1$ on $Ch(A)$. (Indeed, if $x \in Ch(A)$ with $|h(x)|f(x) < \alpha < 1$, let $V = \{w \in \Omega : |h(w)|f(w) > \alpha\}$. By Lemma 3.2, there exists $a \in A_1$ with $a(x) > \alpha$ and $|a| < \frac{\alpha}{\|hf\|}$ on V. Hence $\alpha < \|a\| = \|ahf\| \le \alpha$.)

A similar argument shows that there exists $k \in A^{-1}$ with $|k|f^{-1} \le 1$ on Ω and $|k|f^{-1} = 1$ on $Ch(A)$. Hence $|hk| = 1$ on $Ch(A)$, and consequently on all of Ω. Thus

$$f \le |h^{-1}| = |k| \le f$$

everywhere, so that $f = |k| \in Q$. □

Corollary 3.7. *Assume that Ω is a Shilov boundary of a uniform algebra A, and that f is a strictly positive continuous function on Ω. Then the following conditions are equivalent:*

1. *$f \in Q$;*
2. *A and Af are A-module isometric;*
3. *A and Af are linearly isometric.*

Indeed, if $f_1, f_2 \in C(\partial A)^+$, then $Af_1 \cong Af_2$ A-isometrically, if and only if $f_1 f_2^{-1} \in Q$.

Proof. By Lemma 3.3, $\Omega = \partial A = \partial(Af)$. Now it is clear that Theorem 3.4 gives $(3 \Rightarrow 1)$. A similar argument proves the last statement. The rest of the numbered equivalences are clear. □

The last corollary is not valid without the assumption that Ω is equal to the Shilov boundary of A. Indeed, if we put $A = A(\mathbb{D})$, $\Omega = \bar{\mathbb{D}}$, and define $f(z) = \frac{1+|z|}{2}$, then the map of multiplication by f is an A-module isometry from A onto Af. On the other hand, f is not equal to the absolute value of a function from A.

The first part of the last corollary also follows from

Corollary 3.8. *Suppose that A is a uniform algebra on Ω, and $f \in C(\Omega)^+$. Then $Af \cong A$ A-isometrically if and only if $Af \cong (Af)_{|(\partial A)}$ isometrically via the restriction map, and $f_{|(\partial A)} \in Q(\partial A)$.*

Proof. The ($\Leftarrow$) direction is trivial. The ($\Rightarrow$) direction follows easily from the theorem, but we will give a different proof, which will generalize later to the "almost isometric" case. Assuming that $Af \cong A$ A-isometrically, as in the proof of Corollary 3.6, we see that there exists an $h \in A^{-1}$ such that $\|ahf\| = \|a\|$, and that the latter statement implies that $|(hf)_{|(\partial A)}| = 1$ on ∂A, and $|hf| \leq 1$ on Ω. Moreover, for $w \in \Omega$, we have

$$|(af)(w)| \leq |(ah^{-1})(w)| \leq \|ah^{-1}\|_{\partial A} \leq \|(af)_{|(\partial A)}\| .$$

□

Proof of the Theorem. We will assume that $A = B$ for simplicity, although the same argument works in general. Assume that $T : Af_1 \to Af_2$ is a linear surjective isometry. Let K_i, for $i = 1, 2$, be the set of extreme points of the unit ball in the dual space $(Af_i)^*$. Let $T^*_{|K_2}$ be a restriction of the dual map T^* to K_2. Since T^* is a homeomorphism, in the weak* topology, as well as being an isometry, $T^*_{|K_2}$ is a homeomorphism of K_2 onto K_1. Hence, for $|\alpha| = 1$ and $x \in Ch\,(Af_2)$ we have

$$T^*\,(\alpha x) = \chi\,(\alpha, x) \cdot \varphi\,(\alpha, x)\,,\ \text{where}\ \ |\chi\,(\alpha, x)| = 1 \text{ and } \varphi\,(\alpha, x) \in Ch\,(Af_1)\,.$$

Since T^* is linear, the functions φ and χ depend only on x, and we get the following representation of T:

$$T\,(af_1)\,(x) = \chi\,(x) \cdot (af_1)\,(\varphi\,(x)), \qquad \text{for } a \in A, x \in Ch\,(Af_2)\,,$$

where $\varphi : Ch\,(Af_2) \to Ch(Af_1)$ is a surjective homeomorphism and χ is a unimodular function.

It is easy to see that this implies that φ is continuous on $Ch(Af_2)$. (Indeed, if $x_i \to x_0$ but $\varphi(x_i) \to x_1 \neq \varphi(x_0)$, then choose $a \in A$ with $a(x_1) = 0 \neq a(\varphi(x_0))$. Then $T(af_1)(x_i) \to T(af_1)(x_0)$. However, $|T(af_1)(x_i)| \leq D|a(\varphi(x_i))| \to 0$, but $T(af_1)(x_0) \neq 0$.)

For $a = 1$ we get that $Tf_1\,(x) = \chi\,(x) \cdot f_1(\varphi\,(x)) = f_2(x)\frac{\chi(x)\cdot f_1(\varphi(x))}{f_2(x)}$. Thus the restriction of $\frac{\chi\cdot(f_1\circ\varphi)}{f_2}$ to $Ch(Af_2)$ is the restriction of a function in A. Also $\chi = \frac{Tf_1}{f_1\circ\varphi}$. Hence it follows that χ is also continuous on $Ch(Af_2)$.

Since T is surjective, there is an $a_0 \in A$ such that $T\,(a_0f_1) = f_2 \cdot \left(\frac{\chi\cdot(f_1\circ\varphi)}{f_2}\right)^2$, as functions on $Ch\,(Af_2)$. Hence

$$f_2 \cdot \left(\frac{\chi \cdot (f_1 \circ \varphi)}{f_2}\right)^2 = T\,(a_0f_1) = f_2 \cdot \frac{\chi \cdot ((a_0f_1) \circ \varphi)}{f_2},$$

as functions on $Ch\,(Af_2)$, and so

$$\frac{\chi}{f_2} = (\frac{a_0}{f_1}) \circ \varphi.$$

Consequently we get

$$T\,(af_1)\,(x) = f_2\,(x) \cdot ((a_0a)\,(\varphi\,(x))), \qquad \text{for } x \in Ch\,(Af_2)\,,\, a \in A. \tag{8}$$

Let $b_0 \in A$ be such that $T\,(b_0f_1) = f_2$. We have $f_2 = T\,(b_0f_1) = f_2 \cdot ((a_0b_0) \circ \varphi)$; so $a_0b_0 = 1$ on $Ch\,(Af_1)$. By Lemma 3.3 the closure of $Ch\,(Af_1)$ contains the Shilov boundary of A; so $a_0 b_0 = 1$ on ∂A, and consequently on Ω_1. This proves that a_0 is an invertible element of A. Hence $\{a_0 \cdot a : a \in A\} = A$.

Note that the map $a \longmapsto a \circ \varphi$ is linear, multiplicative, one-to-one, and indeed is isometric since $Ch(A) \subset Ch(Af_1)$. As functions on $Ch(Af_2)$ we have

$$\{a \circ \varphi : a \in A\} = \{(a_0 \cdot a) \circ \varphi : a \in A\} = T(Af_1)/f_2 = A. \tag{9}$$

Thus $a \longmapsto a \circ \varphi$ may be viewed as an isometric automorphism $A \to A$. In particular, $a_0 \circ \varphi$ is an invertible element of A, and we have

$$\frac{f_1 \circ \varphi}{f_2} = \left| \frac{a_0 \circ \varphi}{\chi} \right| = |a_0 \circ \varphi| \in Q,$$

as required.

To finish the proof we need to show that φ can be extended to a homeomorphism of Ω_2 onto a subset of M_A, and that the formula (8) remains valid on the entire set Ω_2. Since $a \longmapsto a \circ \varphi$ is an automorphism of A, it is given by a homeomorphism of the maximal ideal space of the algebra. That is, φ can be extended to a homeomorphism of M_A onto itself mapping ∂A onto ∂A. For a given a in A, $T(af_1)$ and $f_2 \cdot ((a_0 a) \circ \varphi)$ are elements of Af_2 which, by (8), are identical on the Choquet boundary of A. By dividing by f_2 if necessary, we see that these elements are identical at any point of Ω_2, and we get (7). □

Another proof of this result, using the function multiplier algebra, may be found in [8]. However, it is the proof above which extends to the "almost isometric" case.

4. Almost isometries

Recall that Banach spaces X, Y are almost isometric if the Banach-Mazur distance between X and Y, defined as

$$d_{\text{B-M}}(X, Y) = \log \inf \left\{ \|T\| \left\|T^{-1}\right\| : T : X \to Y \right\},$$

is equal to zero; two A-modules X, Y are almost A-isometric if

$$\log \inf \left\{ \|T\| \left\|T^{-1}\right\| : T : X \to Y \right\} = 0,$$

where this time the infimum is taken over the set of all A-isomorphisms. Of course two isometric Banach spaces are almost isometric, but even for the class of separable uniform algebras defined on subsets of a plane, almost isometric spaces need not be isometric [22]. Small bound isomorphisms between various classes of Banach spaces, primarily function spaces, have been investigated in a large number of papers; see for example [2, 23, 24, 32].

Theorem 4.1. *Assume that A is a uniform algebra on a compact set Ω, and that f is a strictly positive continuous function on Ω. Suppose that $0 < \epsilon < \frac{1}{3}$ is given, and suppose that T is a surjective linear map from A onto Af such that $\|T\| \leq 1$ and $\left\|T^{-1}\right\| \leq 1 + \varepsilon$. Then there are a subset Ω_0 of the Shilov boundary of Af and a surjective continuous map $\varphi : \Omega_0 \to Ch(A)$ such that*

$$|T(a)(x) - T(\mathbf{1})(x) \cdot a \circ \varphi(x)| \leq 4\varepsilon \frac{1+\varepsilon}{1-\varepsilon} \|a\|, \quad \text{for } a \in A,\ x \in \Omega_0, \tag{10}$$

and

$$1 - 10\varepsilon \leq |T(\mathbf{1})(x)| \leq 1, \quad \text{for } x \in \Omega_0. \tag{11}$$

It follows that

$$\sup\left\{\left|\left(af\right)(x)\right|:x\in\Omega_0\right\}\geq(1-15\varepsilon)\left\|af\right\|,\quad \text{for } af\in Af, \tag{12}$$

and that the closure of Ω_0 *contains the Shilov boundary of* A.

Proof. Let $\widetilde{T}:A\to Af_{|\partial(Af)}\subset C\left(\partial(Af)\right)$ be defined by $\widetilde{T}\left(a\right)=(1+\epsilon)T\left(a\right)_{|\partial(Af)}$. We know from Lemma 3.1 (iii) that the restriction map from Af to $Af_{|\partial(Af)}$ is an isometry. By Theorem 6.1 of [23] applied to $\widetilde{T}$, there are a subset Ω_0 of $\partial(Af)$ and a continuous function φ from Ω_0 onto $Ch(A)$ such that (10) holds.

Assume now that there is an $x_0\in\Omega_0$ with $\left|T\left(\mathbf{1}\right)(x_0)\right|<1-10\varepsilon$. At the beginning of the proof of Theorem 6.1 in [23], the set Ω_0 is defined specifically as a subset of $\left\{x\in\Omega:\|x\|_{Af}>M\right\}$, where we denote by $\|x\|_{Af}$ the norm of the "evaluation at the point x" functional on Af, and where M can be chosen to be any number satisfying $\frac{1-\varepsilon}{1+\varepsilon}>M>\frac{1-\varepsilon}{1+\varepsilon}-\varepsilon^2$. Since $1-3\varepsilon<\frac{1-\varepsilon}{1+\varepsilon}-\varepsilon^2$, there is a norm-one element a_0f of Af such that $\left(a_0f\right)(x_0)\geq 1-3\varepsilon$. Put $a_1=T^{-1}\left(a_0f\right)$. We have $\|a_1\|\leq\left\|T^{-1}\right\|\leq 1+\varepsilon$ and $T\left(a_1\right)(x_0)\geq 1-3\varepsilon$, while

$$\left|T\left(\mathbf{1}\right)(x_0)\cdot a_1\circ\varphi\left(x_0\right)\right|\leq\left|T\left(\mathbf{1}\right)(x_0)\right|\|a_1\|\leq(1-10\varepsilon)\left(1+\varepsilon\right).$$

This contradicts (10) and shows (11).

To prove (12), assume that there is a norm-one element af of Af such that

$$\sup\left\{\left|af\left(x\right)\right|:x\in\Omega_0\right\}<(1-15\varepsilon).$$

Put $b=T^{-1}\left(af\right)$, let $\widetilde{x}\in Ch(A)$ be such that $\left|b\left(\widetilde{x}\right)\right|=\|b\|\geq 1$, and let $x_1\in\Omega_0$ be such that $\varphi\left(x_1\right)=\widetilde{x}$. By (11) we have that

$$\left|T\left(\mathbf{1}\right)(x_1)\cdot b\circ\varphi\left(x_1\right)\right|\geq 1-10\varepsilon,\text{ while }\left|Tb\left(x_1\right)\right|=\left|af\left(x_1\right)\right|\leq 1-15\varepsilon,$$

which contradicts (10) and shows (12).

To finish the proof we need to show that (12) implies $\partial A\subset\overline{\Omega}_0$. To this end, choose a point $x_0\in Ch(A)$ which is not in $\overline{\Omega}_0$. W.l.o.g. we may assume that $\|f\|\leq 1$. By Lemma 3.2 there is an $a\in A$ such that

$$\|a\|=1=a\left(x_0\right)\text{ and }\left|a\left(x\right)\right|<\frac{1}{2}\min f,\quad\text{for } x\in\overline{\Omega}_0.$$

By (12) we have

$$\begin{aligned}\min f&\leq\left|\left(af\right)(x_0)\right|\leq\|af\|\\&\leq\frac{1}{1-15\varepsilon}\sup\left\{\left|\left(af\right)(x)\right|:x\in\Omega_0\right\}\\&\leq\frac{1}{1-15\varepsilon}\frac{1}{2}\min f.\end{aligned}$$

This contradiction proves that $\partial A\subset\overline{\Omega}_0$. □

Corollary 4.2. *Assume that A is a uniform algebra on a compact set Ω, and that f is a strictly positive continuous function on Ω. The following conditions are equivalent:*

1. *A and Af are almost A-isometric;*

2. *A and Af are almost isometric.*

Moreover, if Ω is equal to the Shilov boundary, the above conditions are also equivalent to

3. $f\in\overline{Q}^+$.

Proof. That (3) implies (1) implies (2) is left to the reader.

Assume that A and Af are almost isometric, and let $T : A \to Af$ be such that $\|T\| \le 1$ and $\left\|T^{-1}\right\| \le 1+\varepsilon$. Since T is surjective and $\frac{T(\mathbf{1})}{f} \in A$, there is an $a_0 \in A$ such that $Ta_0 = \left(\frac{T(\mathbf{1})}{f}\right)^2 f$. By (10) we get

$$\begin{aligned} \left\| \left(\frac{T(\mathbf{1})}{f}\right)^2 f - T(\mathbf{1}) \cdot a_0 \circ \varphi \right\|_{\Omega_0} &\le 4\varepsilon \frac{1+\varepsilon}{1-\varepsilon} \|a_0\| \\ &\le 4\varepsilon \frac{1+\varepsilon}{(1-\varepsilon)^2} \left\| \left(\frac{T(\mathbf{1})}{f}\right)^2 f \right\| \\ &= 4\varepsilon \frac{1+\varepsilon}{(1-\varepsilon)^2} \left\| (T(\mathbf{1}))^2 f^{-1} \right\| \\ &\le 4\varepsilon \frac{1+\varepsilon}{(1-\varepsilon)^2} \left\| f^{-1} \right\|; \end{aligned}$$

so by (11) we have

(13)
$$|T(\mathbf{1})(x) - f(x) \cdot a_0(\varphi(x))| \le 4\varepsilon \frac{1+\varepsilon}{(1-10\varepsilon)(1-\varepsilon)^2} \left\|f^{-1}\right\| \|f\|, \quad \text{for } x \in \Omega_0.$$

From (10) and (13) we obtain

(14) $$|T(a)(x) - f(x) \cdot (aa_0)(\varphi(x))| \le \varepsilon' \|a\|, \quad \text{for } a \in A,\ x \in \Omega_0,$$

where $\varepsilon' = 4\varepsilon \frac{1+\varepsilon}{(1-10\varepsilon)(1-\varepsilon)^2} \left\|f^{-1}\right\| \|f\| + 4\varepsilon \frac{1+\varepsilon}{1-\varepsilon}$.

Let $b_0 \in A$ be such that $Tb_0 = f$. If ε is sufficiently small, then (14) gives

$$\begin{aligned} \|\mathbf{1} - (a_0 b_0) \circ \varphi\|_{\Omega_0} &\le \left\|f^{-1}\right\| \|f - f \cdot ((a_0 b_0) \circ \varphi)\|_{\Omega_0} \\ &= \left\|f^{-1}\right\| \|Tb_0 - f \cdot ((a_0 b_0) \circ \varphi)\|_{\Omega_0} \\ &< \frac{1}{2}\ . \end{aligned}$$

Hence $\operatorname{Re}(a_0 b_0)(x) \ge \frac{1}{2}$ for any $x \in Ch(A)$, and consequently for any x in the maximal ideal space of A. It follows that a_0 and b_0 are invertible in A. By (13) and (14) the function $\frac{T(\mathbf{1})}{f} \cdot \frac{T\left(a_0^{-2}\right)}{f}$ is approximately equal, on Ω_0, to $(a_0 \circ \varphi) \cdot (a_0^{-1} \circ \varphi) = 1$. Since $\bar{\Omega}_0$ contains ∂A, the function $\frac{T(\mathbf{1})}{f} \cdot \frac{T\left(a_0^{-2}\right)}{f} - \mathbf{1}$ is approximately equal to zero on the maximal ideal space of A. Thus, if ε is sufficiently small, $\frac{T(\mathbf{1})}{f} \cdot \frac{T\left(a_0^{-2}\right)}{f}$ is an invertible element of A. Consequently $\frac{T(\mathbf{1})}{f}$ is invertible.

We can now define an A-module isomorphism $S : A \to Af$ by

$$Sa = T(\mathbf{1})\, a, \quad \text{for } a \in A.$$

Fix $a \in A$, and let $\widetilde{a} \in A$ be such that $T\widetilde{a} = Sa = T(\mathbf{1})\, a$. By Theorem 4.1, $T\widetilde{a}$ is close, on Ω_0, to $T(\mathbf{1})\, \widetilde{a} \circ \varphi$. It follows that $a \approx \widetilde{a} \circ \varphi$ on Ω_0. Consequently, by (12) and the fact that T is an "almost isometry", we get $\|a\| \approx \|\widetilde{a} \circ \varphi\| = \|\widetilde{a}\| \approx \|T\widetilde{a}\| = \|Sa\|$. Thus S is also an "almost isometry".

Now assume that Ω is equal to the Shilov boundary. By (11) we have that $|T(\mathbf{1})| \approx 1$ on $\overline{\Omega}_0 = \partial A$. As we proved before, $\frac{T(\mathbf{1})}{f}$ is an invertible element of A. It follows that $f \in \overline{Q}^+$. ☐

We have the following complement to the previous corollary (cf. Corollary 3.8):

Corollary 4.3. *Suppose that A is a uniform algebra on Ω, and that $f \in C(\Omega)^+$. Then $Af \cong A$ almost A-isometrically if and only if $Af \cong (Af)_{|(\partial A)}$ isometrically via the restriction map, and $f_{|(\partial A)} \in \bar{Q}^+(\partial A)$.*

Proof. This proof is almost identical to the proof of Corollary 3.8. As in the proof of Corollary 3.6, it follows that for all $\epsilon > 0$, there exists an $h_\epsilon \in A^{-1}$ such that

$$(1-\epsilon)\|a\| \le \|ah_\epsilon f\| \le (1+\epsilon)\|a\|,$$

and that the latter statement implies that $|(h_\epsilon f)_{|(\partial A)}| \approx 1$ on ∂A, and $|hf| \le 1+\epsilon$ on Ω. Thus $f_{|(\partial A)} \in \bar{Q}^+(\partial A)$. Moreover, for $w \in \Omega$, we have

$$|(af)(w)| \le (1+\epsilon)|(ah_\epsilon^{-1})(w)| \le (1+\epsilon)\|ah_\epsilon^{-1}\|_{\partial A} \le (1+\epsilon)^2\|(af)_{|(\partial A)}\| \,.$$

Since $\epsilon > 0$ is arbitrary, $\|af\| = \|af\|_{(\partial A)}$. ☐

Corollary 4.4. *For any strictly positive $f \in C(\Omega)$, the following are equivalent:*
- (i) *$f \in \bar{Q}^+$;*
- (ii) *$A \cong Af \cong Af^{-1}$ almost A-isometrically;*
- (iii) *$A \cong Af \cong Af^{-1}$ linearly almost isometrically.*

Proof. We will only prove $(ii) \Rightarrow (i)$. That (iii) implies (ii) follows from what we just did. The easy implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are left to the reader.

If $T_1 : A \to Af$ and $T_2 : A \to Af^{-1}$ are module isomorphisms, let $t_1 = T_1(\mathbf{1})$ and $t_2 = T_2(\mathbf{1})$. Then $T_1(a) = t_1 a$, $T_2(a) = t_2 a$, and $t_1 f^{-1}$, $t_2 f$ are invertible elements of A. If $\|T_1\| \le 1+\varepsilon, \|T_2\| \le 1+\varepsilon$, then

$$|t_1(x)| \le 1+\varepsilon \quad \text{and} \quad |t_2(x)| \le 1+\varepsilon, \qquad \text{for } x \in \Omega, \tag{15}$$

and consequently

$$|t_1(x)\, t_2(x)| \le (1+\varepsilon)^2, \qquad \text{for } x \in \Omega.$$

Suppose that $\left\|T_1^{-1}\right\| \le 1, \left\|T_2^{-1}\right\| \le 1$. If $|t_1(x_0)| = 1 - r < 1$ for some $x_0 \in Ch(A)$, then by Lemma 3.2 there is an $a \in A$ such that $a(x_0) = 1 = \|a\|$ and $|a(x)| < \frac{1}{2+\varepsilon}$ for $x \in \left\{x : |t_1(x)| > 1 - \frac{r}{2}\right\}$. We get

$$\|T_1(a)\| = \|t_1 a\| \le \max\left\{\frac{1}{2+\varepsilon}\|T_1\|, 1-\frac{r}{2}\right\} < 1.$$

This contradicts the assumption that $\left\|T_1^{-1}\right\| \le 1$.

Hence

$$|t_1(x)| \ge 1 \quad \text{and} \quad |t_2(x)| \ge 1, \qquad \text{for } x \in Ch(A);$$

so

$$|t_1(x)\, t_2(x)| \ge 1, \qquad \text{for } x \in Ch(A).$$

However, $t_1 t_2$ is invertible; so it attains its minimum on $Ch(A)$. Hence

$$1 \le |t_1(x)\, t_2(x)| \le (1+\varepsilon)^2, \qquad \text{for } x \in \Omega.$$

By (15) it follows that

$$\frac{1}{1+\varepsilon} \leq |t_2(x)| \leq 1+\varepsilon, \quad \text{for } x \in \Omega.$$

Thus $\|f - |t_2| f\| \leq \varepsilon \|f\|$, and since $\varepsilon > 0$ is arbitrary we get $f \in \bar{Q}^+$. □

Lemma 4.5. *If X is a Banach A-module over a function algebra A, and if X is almost A-isometric to a function A-module, then X is a function A-module.*

Proof. We use the injective tensor norm characterization of function modules [9]. Suppose that T_ϵ is the ϵ-isomorphism. Then for $a_1, \cdots, a_n \in A$ and $x_1, \cdots, x_n \in X$ we have

$$\|\sum_i a_i T_\epsilon(x_i)\| \leq \|\sum_i a_i \otimes T_\epsilon(x_i)\|_\lambda \leq \|T_\epsilon\| \|\sum_i a_i \otimes x_i\|_\lambda \ ,$$

where λ is the injective tensor norm. Thus

$$\|\sum_i a_i x_i\| = \lim_{\epsilon \to 0} \|T_\epsilon(\sum_i a_i x_i)\| \leq \|\sum_i a_i \otimes x_i\|_\lambda \ .$$

By [9], X is a function A-module. □

Corollary 4.6. *Let X be a Banach A-module. The following are equivalent:*

(i) *X is an algebraically singly generated, faithful function A-module, and $X \cong A$ almost linearly isometrically;*
(ii) *$X \cong A$ almost A-isometrically;*
(iii) *there exists $f \in \bar{Q}^+$ such that $X \cong Af$ A-isometrically.*

Proof. Assume (i). As we said early in the introduction, the first two conditions in (i), together with a corollary in §3 of [8], show that $X \cong Af$ A-isometrically, where $f \in C(\Omega)^+$ for some Ω on which A acts as a function algebra. Now (ii) follows from Corollary 4.2. By Corollary 4.3, $Af \cong (Af)_{|(\partial A)}$ and $f_{|(\partial A)} \in \bar{Q}^+(\partial A)$, showing (iii).

Given (ii), it follows by Lemma 4.5 that X is a function A-module, and now (i) is clear. Clearly (iii) implies (i). □

A similar result (and proof) holds with almost isometries replaced by isometries.

We notice that the results in this section have been stated for a single function f, rather than for a pair of functions f_1, f_2, like results in the previous section concerning isometries. There are analogous results describing almost isometries between modules Af_1 and Af_2; however, they are much more technical and involve not a single automorphism $\varphi : \Omega \to \Omega$ but a sequence of homeomorphisms between the Choquet boundaries of Af_1 and Af_2. The following theorem, which may be of independent interest, can be proven using methods similar to that of the proof of Theorem 6.1 of [23]. It may then be used to extend the last few results to almost isometries between modules Af_1 and Af_2.

Theorem 4.7. *Assume that Ω is the Shilov boundary of a uniform algebra A, and that f_1, f_2 are strictly positive continuous functions on Ω. Suppose that $T : Af_1 \to Af_2$ is a surjective linear isomorphism such that $\|T\| \leq 1+\varepsilon$ and $\|T^{-1}\| \leq 1+\varepsilon$, where $\varepsilon < \varepsilon_0$ (an absolute constant). Then there are a dense subset S of Ω containing the Choquet boundary of A, a continuous bijection $\varphi : S \to Ch(A)$, and a continuous unimodular function χ such that*

$$\|T(af_1) - \chi \cdot (af_1) \circ \varphi\| \leq \varepsilon' \|af_1\|, \quad \textit{for } a \in A, \tag{16}$$

where $\varepsilon' \to 0$ *as* $\varepsilon \to 0$.

If the functions $f_1, f_2 \in \mathcal{M}_A$, then a simple version of an "almost isometry" result may be found in Theorem 7.6.

No doubt there are also versions of all these results for a pair of function algebras A and B, and function modules Af_1 and Bf_2, but that would take us even further afield from our main concerns.

Part B

5. Some observations concerning approximations in modulus

In this section we make various observations concerning the sets $P, Q, \mathcal{G}, \mathcal{F}$, $GH_A(\Omega), H_A(\Omega)$ and $\mathcal{M}_A(\Omega)$ defined in the Introduction, Notation and Definitions sections. These sets will play a crucial role in the next sections when we study singly generated bimodules.

We already observed in §2 that $C(\Omega)^+ = \bar{Q}^+$ if and only if A is logmodular on Ω. The question of when $C(\Omega)_+ = \bar{P}$ (resp. $C(\Omega)_+ = \mathcal{G}$) has been studied by Mlak, Glicksberg, Douglas and Paulsen, and others. An algebra with this property is called "approximating in modulus" (resp. "convexly approximating in modulus"). Just as in the usual proof that A may only be logmodular on ∂A, one can show, using Urysohn's lemma, Lemma 3.2, and the fact that a function in A has maximum modulus achieved on ∂A, that either of these "approximating in modulus" properties forces $\Omega = \partial A$.

For example, Dirichlet algebras, such as the disk algebra $A(\mathbb{D})$ considered as functions on the circle, are logmodular, and approximating in modulus. Glicksberg gave the following sufficient condition: If the inner functions (that is, functions in A which have constant modulus 1 on Ω) separate points of Ω, then A is approximating in modulus.

By the Stone-Weierstrass theorem it is easy to see that for any function algebra A on Ω the set $\mathcal{F} - \mathcal{F}$ is dense in $C_{\mathbb{R}}(\Omega)$. However, this does not imply that $\mathcal{F}$ is dense in $C(\Omega)_+$, or equivalently, that A is convexly approximating in modulus. Indeed, we have the following:

Proposition 5.1. *Let A be a function algebra. If $Ch(A) \neq \partial A$, then A is not convexly approximating in modulus on ∂A. That is, $\mathcal{G} \neq C\left(\partial A\right)_+$.*

Proof. Let $x_0 \in \partial A$, let $V \subset \partial A$ be an open neighborhood of x_0, and let $f \in C\left(\partial A\right)_+$ be such that

$$f\left(x_0\right) = 1 = \|f\|\,, \text{ and } f\left(x\right) = 0 \text{ for } x \in \partial A - V.$$

Assume that $\mathcal{G} = C\left(\partial A\right)_+$, and let $g_1, ..., g_n \in A$ be such that

$$\left| f - \sum_{j=1}^{n} \left|g_j\right|^2 \right| \leq \frac{1}{4}.$$

Multiplying $g_1, ..., g_n$ by appropriate numbers of absolute value 1, we may assume that $g_j\left(x_0\right) \in \mathbb{R}_+$. Then

$$\frac{3}{4} \leq \left(\sum_{j=1}^{n} \left|g_j\right|^2 \right)\left(x_0\right) = \sum_{j=1}^{n} g_j\left(x_0\right)^2 \leq \frac{5}{4}.$$

Put

$$g = \frac{\sum_{j=1}^{n} g_j^2}{\sum_{j=1}^{n} g_j\left(x_0\right)^2} \in A.$$

We have

$$g\left(x_0\right) = 1, \; \|g\| \leq \frac{5}{3}, \text{ and } \left|g\left(x\right)\right| \leq \frac{1}{3} \text{ for } x \in \partial A - V.$$

By the Bishop "$\frac{1}{4} - \frac{3}{4}$" criterion ([17], Th. 11.1, page 52, and remark on p. 59) x_0 is a p-point of A. That is, $x_0 \in Ch(A)$. □

Lemma 5.2. *Let A be a function algebra, and suppose that $f^2 \in \mathcal{G}(\Omega)$. Then f achieves its norm on ∂A. Indeed, $Af \cong (Af)_{|(\partial A)}$ A-isometrically.*

Proof. Suppose that K_n are A-tuples with $\|K_n(\cdot)\|_2$ converging uniformly to f. Given $\epsilon > 0$, we have $\|K_n(w)\|_2 \leq f(w) + \epsilon$ for all $w \in \Omega$ and sufficiently large n. For such w, n, and for any complex Euclidean unit vector z, and $a \in A$, we have

$$|z.(a(w)K_n(w))| \leq \|z.(a(\cdot)K_n(\cdot))\|_{\partial A} \leq \|af\|_{\partial A} + \epsilon\|a\| \;\; .$$

Thus $\|a(w)K_n(w)\|_2 \leq \|af\|_{\partial A} + \epsilon\|a\|$. Letting $n \to \infty$ gives $\|af\|_\Omega \leq \|af\|_{\partial A} + \epsilon\|a\|$. Since $\epsilon > 0$ was arbitrary, we get the result. □

Using this, we can sharpen an earlier result:

Corollary 5.3. *Let A be a function algebra. Suppose that $f \in C(\Omega)^+$, that $f^{-2} \in \mathcal{G}(\Omega)$, and that $Af \cong A$ A-isometrically (resp. almost A-isometrically). Then $f \in Q(\Omega)$ (resp. $f \in \bar{Q}^+(\Omega)$).*

Proof. By Lemma 5.2 and Corollary 3.8 (resp. 4.3), $f_{|(\partial A)} \in Q(\partial A)$ (resp. in $\bar{Q}^+(\partial A)$), and $Af^{-1} \cong (Af^{-1})_{|(\partial A)} \cong A$. Now use Corollary 3.6 (resp. 4.4). □

Note that for the disk algebra $A(\mathbb{D})$ considered as functions on the closed disk, or more generally for any function algebra containing no nontrivial inner functions, it is clear that $\{f : f, f^{-1} \in P\} = Q$. Thus for certain good examples, we will have to look among algebras with many inner functions. It is also easy to see that this relation is also true if $\Omega = M_A$.

It will be significant to us that $Q, \bar{Q}^+$ and $\mathcal{M}$ are (abelian) groups. We will write Q' for the quotient group $\bar{Q}^+/Q$. Of course, for some function algebras, it can happen that $Q = \bar{Q}^+$, for example for $H^\infty(\mathbb{D})$, where $\bar{Q}^+ = L^\infty(\mathbb{T})^+$ ([35], Th. 5.26). On the other hand, for $A(\mathbb{D})$, the disk algebra, we have that Q is a proper subset of $\bar{Q}^+ = C(\mathbb{T})^+$.

A good way of producing interesting functions in $\bar{Q}^+$, which works in any nonselfadjoint function algebra, goes as follows: By a result of Hoffman and Wermer [34], $\mathrm{Re}\, A$ is not uniformly closed. If $g \in \overline{(\mathrm{Re}\;\, A)} \setminus \mathrm{Re}\, A$, set $f = e^g$. If $a_n \in A$ and $\mathrm{Re}\, a_n \to g$ uniformly, then $e^{\mathrm{Re}\;\, a_n} = |e^{a_n}| \to f$. Hence $f \in \bar{Q}^+$. If the set of invertible elements in A is connected, or equivalently ([17], p. 91), if the first Čech cohomology group $H^1(M_A, \mathbb{Z}) = 0$ of the maximal ideal space M_A is zero, then $h \in Q$ if and only if $h = e^{\mathrm{Re}\;\, a}$ for some $a \in A$; thus we can definitely assert that $f \notin Q$ in this case. Thus we have proved:

Corollary 5.4. *Suppose that A is a nonselfadjoint function algebra such that $H^1(M_A, \mathbb{Z}) = 0$. Then $Q' = \bar{Q}^+/Q \neq 0$. Hence there exist nontrivial function A-modules which are almost A-isometric to A.*

We shall see later that this corollary also gives the existence of nontrivial rank 1 strong Morita equivalence bimodules for any such A.

We now turn to the class $\mathcal{M}_A(\Omega)$ defined in the introduction. We have

$$GH_A(\Omega) \subset H_A(\Omega) \subset \log \mathcal{M}_A(\Omega).$$

In Part C we shall see that all three of the above classes have the five properties of "harmonic classes' with respect to A described in the introduction. We shall not use the following, but state it because it is interesting. Its proof follows from the definition of $\mathcal{M}$ in the introduction, and is left to the reader.

Proposition 5.5. *If $f \in \mathcal{M}_A(\Omega)$ and if $t > 0$, then $tf \in \mathcal{M}$. Moreover, $\mathcal{M}$ is closed in the relative topology from $C(\Omega)^+$. Thus $\log \mathcal{M}_A(\Omega)$ is uniformly closed.*

Note that $\mathcal{M}$ is not complete in the norm topology (since if $f \in \mathcal{M}$ and $t > 0$, then $tf \in \mathcal{M}$, but $\lim_{t\to 0} tf = 0 \notin \mathcal{M}$).

The authors do not know if $H_A(\Omega) = \log \mathcal{M}_A(\Omega)$ in general. In any case, either answer to the question seems very interesting. If $H(\Omega) = \log \mathcal{M}_A(\Omega)$, then we obtain, from what we do later, amongst other things, a neat description of all topologically singly generated Morita equivalence A-A-bimodules; but if $H(\Omega) \neq \log \mathcal{M}_A(\Omega)$ in general, then $\log \mathcal{M}_A(\Omega)$ seems to be a genuinely new and interesting class of harmonic functions with respect to A. A uniform algebra is called "Dirichlet" if $GH_A(\Omega) = C_{\mathbb{R}}(\Omega)$, and is called "logmodular" if $H_A(\Omega) = C_{\mathbb{R}}(\Omega)$; therefore we will call an algebra *logMorita* if $\log \mathcal{M}_A(\Omega) = C_{\mathbb{R}}(\Omega)$. Every logmodular algebra is logMorita. The next result shows that logMorita algebras share many properties with logmodular algebras.

Theorem 5.6. *If A is a logMorita function algebra on Ω, then:*

(i) *Ω is the Shilov boundary of A.*

(ii) *Every $\phi \in M_A$ has a unique representing measure.*

(iii) *If Π is a Gleason part for A, then Π is either a singleton, or an analytic disk, in the sense that there is a bijective continuous map $\Phi : \mathbb{D} \to \Pi$ such that if $f \in A$ then $\hat{f} \circ \Phi$ is holomorphic on $\mathbb{D}$.*

Proof. (i) is proved as for Dirichlet and logmodular algebras, using Urysohn's lemma and the fact that for an A-tuple $H \in A^{(n)}$, the function $\|H(\cdot)\|_2$ achieves its maximum modulus on the Shilov boundary. (The latter fact may be seen by considering $z.H(w)$ for $z \in \mathbb{C}^n_1$.) Similarly, (ii) follows the classical line of proof: Suppose that μ, ν are representing measures for $\phi \in M_A$, and that H, K are A-tuples with $1 = H(w) \cdot K(w)$ for all $w \in \Omega$. By Fubini and Cauchy-Schwarz, we have

$$1 = \hat{H}(\phi) \cdot \hat{K}(\phi) = \int_{\Omega\times\Omega} H(w) \cdot K(z)\, d(\mu \times \nu)$$
$$\leq (\int \|H(w)\|_2 d\mu)(\int \|K(w)\|_2 d\nu)\ .$$

The remainder of the proof follows 17.1 in [34]. Finally, (iii) follows from (ii) by 17.1 of [34]. ☐

It follows from (iii) of the last theorem that, just as for Dirichlet and logmodular algebras, there are no interesting examples of logMorita algebras of holomorphic functions on domains in $\mathbb{C}^n$ for $n > 1$. For suppose that Ω is such a domain, with w an interior point, suppose that B is an open ball centered at w on which every function f in A is holomorphic, and suppose that $z \in B$. Then by restricting A to the piece of the complex line through w and z inside B, we have that the ensuing functions are analytic in the 1-variable sense, and so there is a constant c with $|f(w) - f(z)| \leq c < 2$ for all $f \in Ball(A)$. Thus w and z are in the same Gleason part of A. Therefore this Gleason part contains B, contradicting (iii) of the previous theorem unless $n = 1$.

6. Approximations in modulus and equivalence bimodules

In [11] the notion of *strong Morita equivalence* is defined for a pair of operator algebras A and B. Its theory and consequences have been worked out there and in other papers of ours (see [10, 7] for example). A related notion, *strong subequivalence*, was recently defined in [7]. It was shown to have many of the properties of strong Morita equivalence. One of our objectives here is to show that it is not the same as strong Morita equivalence. It will not be necessary for us to state the general definitions of these notions here; we will simply say that they involve a pair of bimodules X and Y, called *equivalence bimodules*. In fact, we shall restrict our attention here to the special case where the operator algebras are function algebras, and in this case the two definitions can be simplified. Indeed, for a bimodule of the form Af, where f is a strictly positive function on Ω, and considering the canonical pairing $Af \times Af^{-1} \to A$, one may translate the definitions from [11, 7], using Lemma 2.8 of [11], into the following precise form. Since, as the referee pointed out, the translation yielding (ii) is not that easy, we will include the details at the end of this section.

Definition 6.1. Suppose that A is a function algebra on a compact space Ω, and f is a strictly positive continuous function on Ω.

- (i) We say that Af is a *strong Morita equivalence bimodule (with inverse bimodule Af^{-1})*, if whenever $\epsilon > 0$ is given, then we can write $1 = \sum_{i=1}^{n} x_i y_i$ as functions on Ω, with $x_i \in Af, y_i \in Af^{-1}$, and $\sum_{i=1}^{n} |x_i(\omega)|^2 \leq 1 + \epsilon$, and $\sum_{i=1}^{n} |y_i(\omega)|^2 \leq 1 + \epsilon$, for all $\omega \in \Omega$. We will say Af is rank 1, if $n = 1$ in the above.
- (ii) We say that Af is a *strong subequivalence bimodule* if whenever $\epsilon > 0$ is given, there are $x_i \in Af, y_i \in Af^{-1}$, such that $1 - \epsilon \leq \sum_{i=1}^{n} |x_i(\omega)|^2 \leq 1 + \epsilon$, and $1 - \epsilon \leq \sum_{i=1}^{n} |y_i(\omega)|^2 \leq 1 + \epsilon$, for all $\omega \in \Omega$.
- (iii) We say that Af is a *unitary subequivalence bimodule* if (ii) holds, but with $\epsilon = 0$ and $n = 1$.
- (iv) We shall say that Af is a *Shilov subequivalence bimodule* if it is a strong subequivalence bimodule and Ω is the Shilov boundary of A.

All these definitions are in [11, 7] except (iii). Also, strictly speaking, in (i) we should say that $(A, A, Af, Af^{-1}, \cdot, \cdot)$ *is a strong Morita context* (see [11], Definition 3.1). Here "$\cdot$" refers to multiplication of scalar functions on Ω. However, to avoid this somewhat cumbersome notation, we will use the looser convention of (i). Similarly, saying in (ii) that "Af is a strong subequivalence bimodule" is shorthand for something like the following: in the language of [7], Definition 5.2, (A, A, Af, Af^{-1}) is a dilatable subcontext of $(C(\Omega), C(\Omega), C(\Omega), C(\Omega))$. This matter is cleared up completely at the end of this section. Also, concerning (iv), we used the word "minimal" instead of "Shilov" in [7], Definition 5.7.

It is clear, in the definitions above, that (iii) $\Rightarrow$ (ii) and that (i) $\Rightarrow$ (ii).

Proposition 6.2. *Suppose that $f \in C(\Omega)^+$. Then:*

(a) *Af is a unitary subequivalence bimodule if and only if $f \in P \cap P^{-1}$, that is, iff $f \in P$ and $f^{-1} \in P$.*
(b) *Af is a strong subequivalence bimodule if and only if f^2 and f^{-2} are in $\mathcal{G}$.*
(c) *$f \in \bar{Q}^+$ if and only if Af is a rank 1 strong Morita equivalence bimodule (with inverse bimodule Af^{-1}).*
(d) *$f \in \mathcal{M}(\Omega)$ if and only if Af is a strong Morita equivalence bimodule (with inverse bimodule Af^{-1}).*

Proof. (a) Clearly Af is a unitary subequivalence bimodule if and only if Af and Af^{-1} both contain inner functions (functions of constant modulus 1 on Ω), which is clearly equivalent to $f \in P \cap P^{-1}$.

(b) Suppose that $\epsilon > 0$ is given, and that x_i, y_i are as in (ii). Then

$$\|1 - \sum_i |x_i|^2\|_\infty \to 0$$

as one allows $\epsilon \to 0$. Hence $f^{-2} \in \mathcal{G}$. Similarly $f^2 \in \mathcal{G}$. For the converse, notice that the argument is reversible.

(c) If $|g_m| \to f$ uniformly as $m \to \infty$, where $g_m, g_m^{-1} \in A$, then we may write $1 = (g_m^{-1} f)(g_m f^{-1})$, and clearly $\|g_m^{-1} f\|_\infty \to 1$ and $\|g_m f^{-1}\|_\infty \to 1$. This obviously implies the condition in (i) of Definition 6.1 with $n = 1$. Also, this argument is reversible.

(d) This is obvious.

□

Remark. To any reader familiar with notions in [7], we remark that the proof of (b) shows the following. Consider $W = C(\Omega)$ as a module over itself. It is clearly generated as a $C(\Omega)$-module by the A-submodule Af. The proof of (b), together with the argument at the end of this section, shows, in the language of [7], that W is the *C^*-dilation* of Af if and only if $f^{-2} \in \mathcal{G}$.

Theorem 6.3. *Suppose that A is a function algebra which is antisymmetric on Ω, or for which the number of pieces in the antisymmetric decomposition*[2] *of A is finite. Suppose that f is a function such that Af satisfies Definition 6.1* (i) *with $\epsilon = 0$. Then $\log f \in H(\Omega)$, and indeed $f \in Q$.*

Proof. First suppose that A is antisymmetric on Ω. By hypothesis, we have $f = \|K(\cdot)\|_2$ and $f^{-1} = \|H(\cdot)\|_2$, for A-tuples H, K with $1 = H.K = (Hf).(Kf^{-1})$. By the converse to Cauchy-Schwarz, $Hf = K^* f^{-1}$, where "$*$" is the complex conjugate. Thus $h_i k_i = |k_i|^2 f^{-2} \geq 0$. By antisymmetry, $h_i k_i = c_i$, a nonnegative constant. Therefore, if $c_1 \neq 0$, then $h_1, k_1 \in A^{-1}$. We have $c_1 = h_1 k_1 = |k_1|^2 f^{-2}$, so that $f = \frac{1}{\sqrt{c_1}} |k_1| \in Q$.

If Ω is a disjoint union of a finite number of antisymmetric pieces Ω_i, then each Ω_i is open and compact, and by the first part we have $f_{|\Omega_i} = |b_i|$, for an invertible $b_i \in A_{|\Omega_i}$. Put $b(x) = b_i(x)$ if $x \in \Omega_i$; then $b \in C(\Omega)$. It follows from [34], Theorem 12.1 say, that $b \in A$ and $b^{-1} \in A$. Thus $f = |b| \in Q$. □

Comparing 6.2 (c) and 3.7 shows that for any $f \in \bar{Q}^+ \setminus Q$, provided say by 5.4, we have that Af is a nontrivial strong Morita equivalence bimodule.

[2] See [34], Theorem 12.1, for example. An algebra with finite antisymmetric decomposition is a direct sum of finitely many antisymmetric uniform algebras.

From the next theorem together with Proposition 6.2, we will be able to give examples $A_0 f$ of strong subequivalence bimodules which are not strong Morita equivalence bimodules.

Theorem 6.4. *Let A be a uniform algebra on Ω. Suppose that w_1, w_2 are distinct points of Ω, and that G is an invertible element of A such that $G(w_1) = 1 = -G(w_2)$. Put $A_0 = \{a \in A : a(w_1) = a(w_2)\}$, and set $f = |G|$. Then $f \in \mathcal{M}_{A_0}$ if and only if w_1, w_2 are in different Gleason parts of A, and if and only if $f \in \bar{Q}^+_{A_0}$.*

Proof. Assume that $f \in \mathcal{M}_{A_0}$. We will show that w_1, w_2 are in different Gleason parts of A. In the definition in the introduction of $\mathcal{M}_{A_0}$, take $\varepsilon = \frac{1}{m}$ for a natural number m, and choose the corresponding $H_m = [h_1^m \cdots h_{n_m}^m], K_m = [k_1^m \cdots k_{n_m}^m] \in A^{n_m}$ with $\langle H|K^*\rangle = \sum_{i=1}^{n_m} h_i^m k_i^m = 1$. By that definition we may assume that $\|K_m(w)\|_2 \le c_m f(w)$ and $\|H_m(w)\|_2 f(w) \le c_m$ for all $w \in \Omega$. Here c_m is a sequence of real numbers decreasing to 1. Multiplying the functions h_i^m, k_i^m by constants with absolute value one, we may also assume that

$$h_i^m(w_2) \ge 0, \text{ for all } i = 1, ..., n_m. \tag{17}$$

Write $\Theta_m(w) = \frac{K_m(w)}{G(w)}$, and $\Pi_m(w) = H_m(w)G(w)$. So $\|\Theta_m(w)\|_2 \le c_m$ and $\|\Pi_m(w)\|_2 \le c_m$. By the Cauchy-Schwarz inequality we get

$$1 = |\sum_{i=1}^{n_m} h_i^m k_i^m| \le \|\Theta_m(w)\|_2 \|\Pi_m(w)\|_2 \le c_m \|\Theta_m(w)\|_2.$$

Hence

$$\frac{1}{c_m} \le \|\Theta_m(w)\|_2 \le c_m, \text{ for all } w \in \Omega.$$

Obviously, the same formula holds with Θ_m replaced by Π_m. Expanding out the following square as an inner product and using (17), we get

$$\|\Theta_m(w_2) - \Pi_m(w_2)\|_2^2 = \|\Theta_m(w_2) - \Pi_m(w_2)^*\|_2^2 \le 2c_m^2 - 2 \to 0.$$

Hence, by the Pythagorean identity

$$\|\Theta_m(w_2) + \Pi_m(w_2)\|_2 \to 2 \quad \text{as } m \to \infty.$$

Put $W_n = \frac{1}{2}(\Theta_m + \Pi_m)$. Since $h_i^m, k_i^m \in A_0$ and $G(w_1) = 1 = -G(w_2)$, we get

$$\|W_m(w_2) - W_m(w_1)\|_2 = \|2(W_m)(w_2)\|_2 \to 2 \quad \text{as } m \to \infty.$$

On the other hand, for any Euclidean vector z of norm 1 we have

$$\begin{aligned} |(z.W_m)(w_2) - (z.W_m)(w_1)| &\le \|w_2 - w_1\| \, \|z.W_m\| \\ &\le \|w_2 - w_1\| \sup\{\|W_m(w)\|_2 : w \in \Omega\}, \end{aligned}$$

where $\|w_2 - w_1\|$ is the norm of $w_2 - w_1$ considered as a functional on A. From the above it follows that $|(z.W_m)(w_2) - (z.W_m)(w_1)| \le \|w_2 - w_1\| c_m$. Thus

$$\|W_m(w_2) - W_m(w_1)\|_2 \le \|w_2 - w_1\| c_m .$$

Since $c_m \to 1$, we get $\|w_2 - w_1\| = 2$. Thus w_1, w_2 lie in different Gleason parts of A.

To prove the other implication, assume now that w_1, w_2 are in different Gleason parts of A. Choose a sequence of functions r_n analytic in the disc $\bar{\mathbb{D}} = \{w \in \mathbb{C} : |w| \le 1\}$ with $r_n(1) = 1, r_n(-1) = -1$, and $1 - 1/n < |r_n(w)| < 1 + 1/n$ for all $w \in \mathbb{D}$. Such functions can be found by taking a conformal equivalence of

$\mathbb{D}$ with a "smile shaped region" inside the annulus $1-1/2n<|w|<1+1/2n$, with the two tips of the smile at -1 and 1. Since w_1, w_2 are in different Gleason parts of A, for any $n \in \mathbb{N}$ large enough, there is an a_n in A such that $\|a_n\| \leq 1-\frac{1}{2n}$, $\left|(r_n \circ a_n)(w_1) - \left(1-\frac{1}{2n}\right)\right| < \frac{1}{2n}$, $\left|(r_n \circ a_n)(w_2) + \left(1-\frac{1}{2n}\right)\right| < \frac{1}{2n}$. Also the norm, on $\{a \in A : a(w_1)=0\}$, of the "evaluation at w_2" functional is equal to one. Thus there is an $\widetilde{a} \in A$ such that $\|\widetilde{a}\| \leq \frac{1}{n}$, $\widetilde{a}(w_1) = 0$ and $\widetilde{a}(w_2) = (r_n \circ a_n)(w_1) + (r_n \circ a_n)(w_2)$. Put $b_n = r_n \circ a_n - \widetilde{a} \in A$. Notice that $b_n(w_1) G(w_1) = b_n(w_2) G(w_2)$; so $b_n G \in A_0$. Moreover, the spectrum of b_n is contained inside the annulus $1-2/n<|w|<1+2/n$. Hence b_n is an invertible element of A, and consequently $b_n G$ is an invertible element of A_0. Thus $f=\lim |b_n G| \in \bar{Q}_{A_0} \subset \mathcal{M}_{A_0}$. □

Example. Let A be a function algebra. First take $\Omega = M_A$ and suppose that A_0, G, f and w_1, w_2 are as in the last theorem. Now, for any Ω on which A sits as a function algebra (such as ∂A), suppose further that there exists a function $h \in A$ such that $|h|=1$ on Ω and $h(w_1) = -h(w_2)$. We are not assuming $w_1, w_2 \in \Omega$ here; they are points in M_A. Then on Ω, $f=|G|=|hG| \in P_{A_0}(\Omega)$, and $f^{-1} = |G^{-1}| = |\frac{h}{G}| \in P_{A_0}(\Omega)$. Thus by 6.2 the submodule $X = A_0 f$ of $C(\Omega)$ is a strong subequivalence A_0-A_0-bimodule, indeed a unitary subequivalence bimodule. However, if w_1, w_2 are in the same Gleason part of M_A, then a simple variant of the first part of the proof above shows that X is not a strong Morita equivalence A_0-A_0-bimodule.

For a very concrete example, let $A(\mathbb{D})$ for the disk algebra, and set $\Omega = \mathbb{T}$, the unit circle. Choose two points $\alpha, \beta \in \mathbb{D}$. Then there exists an inner function h in $A(\mathbb{D})$ such that $h(\alpha) = -h(\beta)$. For example, if $\alpha = -\beta = \frac{1}{2}$, let $h(z) = z$. Choose $G \in A(\mathbb{D})^{-1}$ such that $G(\alpha) = -G(\beta)$. Then the submodule $A_0|G|$ of $C(\mathbb{T})$ is a strong subequivalence bimodule which is not a strong Morita equivalence A_0-A_0-bimodule.

Question. Suppose Af is a subequivalence bimodule, where Ω is the maximal ideal space of A. Then is Af a strong Morita equivalence bimodule? In other words, is $\{f : f^2, f^{-2} \in \mathcal{G}(M_A)\} = \mathcal{M}$? Recall that $\{f : f, f^{-1} \in P(M_A)\} = Q(M_A)$. By the way, it follows easily from this latter fact, and the example above, that "$P \cap P^{-1}$" is not a harmonic class.

In the spirit of the calculation in the proof above, we give some alternative descriptions of $\mathcal{M}_A(\Omega)$.

Proposition 6.5. *For a uniform algebra A on a compact set Ω, the class $\mathcal{M}_A(\Omega)$ coincides with the set of those functions $f \in C(\Omega)^+$ for which there exist a sequence of positive integers $\{n_m\}$ and a sequence of A-tuples $H_m, K_m \in A^{n_m}$, such that the three limits*

(i) $\|K_m(w)\|_2 \to f(w)$,
(ii) $\|H_m(w)\|_2 \to f(w)^{-1}$, *and*
(iii) $\|K_m(w) - H_m(w)^* f(w)^2\|_2 \to 0$

are valid uniformly over $w \in \Omega$. Here the "$*$" represents the complex conjugate of the vector.

Proof. Suppose that $f \in \mathcal{M}_A(\Omega)$. As in the proof of the previous theorem, take $\epsilon = \frac{1}{m}$ for a natural number m, and choose K_m, H_m, c_m as in that proof. Write

$\Theta_m(w) = \frac{K_m(w)}{f(w)}$ and $\Pi_m(w) = H_m(w)f(w)$. So $\|\Theta_m(w)\|_2 \leq c_m$ and $\|\Pi_m(w)\|_2 \leq c_m$. Again by Cauchy-Schwarz we get

$$\frac{1}{c_m} \leq \|\Theta_m(w)\|_2 \leq c_m$$

for all $w \in \Omega$; and the same formula holds for Π_m. By expanding out the following square as an inner product, we see again that

$$\|\Theta_m(w) - \Pi_m(w)^*\|^2 \leq 2c_m^2 - 2 \to 0$$

uniformly as $m \to \infty$.

This gives one direction of the proposition. However, the argument is reversible until at the end we obtain that $H_m(w) \cdot K_m(w) \to 1$ uniformly in $w \in \Omega$. Thus $b_m = H_m(w) \cdot K_m(w)$ is invertible in A, and $(b_m^{-1} H_m(w)) \cdot K_m(w) = 1$. □

Notice that (i) and (ii) in the proposition say exactly that Af is a strong subequivalence bimodule, or that $f^2, f^{-2} \in \mathcal{G}$. Also, notice in fact that (iii), together with either of (i) or (ii), implies the other condition. However, it is convenient to state it as such.

The last proposition may be loosely phrased as saying that a function $f \in C(\Omega)^+$ is in $\mathcal{M}$ if and only if there exists a sequence of A-tuples K_m such that (i) holds uniformly, and $\frac{K_m(\cdot)^*}{\|K_m(\cdot)\|_2^2})$ is "eventually, uniformly, an A-tuple".

In view of (i) and (ii) we may replace (iii) by the more appealing-looking condition

(iii)′ $\|K_m'(w) - H_m'(w)^*\|_2 \to 0$

uniformly on Ω, where $v' = \frac{v}{\|v\|}$ is the unit vector in the same direction as v, for any vector v in complex Euclidean space.

Finally, we end this section by discussing how "strong subequivalence" reduces to the condition in 6.1 (ii) and vice versa. The referee of this paper pointed out that this was not clear. The remainder of this section, added in September '01, hopefully completely clarifies this issue. In fact the argument is quite general and seems to apply equally well to the noncommutative/general operator algebra case too, with little extra effort. Thus we include the more general result.

Our first remark is that in 6.1 (ii), when we say that "Af is a strong subequivalence bimodule", we must mean just a little more than the definition given in [7], Definition 5.2 (and the statement just after it). For that definition does not quite make sense without a given "containing C^*-context" in the language of [7], section 5. One's first thought is to then define "Af is a strong subequivalence bimodule" to mean that there exists a C^*-context $(\mathcal{C}, \mathcal{D}, W, Z)$, where $\mathcal{C}$ is a C^*-algebra generated by A, and that there exists a completely isometric left A-module map $\psi : Af \to W$, with range X say, such that (A, B, X, Y) is a strong subequivalence context in $(\mathcal{C}, \mathcal{D}, W, Z)$, for some subspaces B and Y of $\mathcal{D}$ and Z respectively. However, with a little thought one sees that this definition is a little problematic in some ways, and in particular is not literally equivalent to 6.1 (ii), unless one imposes some extra reasonable hypothesis. (We remark that insisting ψ be a bimodule map does not help. We also remark in passing that if A is a function algebra on Ω, and if Af is a "strong subequivalence bimodule" in the sense above (without any extra hypothesis), and if one is content to replace Af by a module Af' which is A-isometric to Af, with $f' \in C(\Omega)^+$, then what we do below will show that we can take $X = Af'$ to satisfy the conditions in 6.1 (ii).)

There are two such extra hypotheses that we can show to work: (a) to insist, in addition to the conditions in the third sentence of the last paragraph, that we have $(\mathcal{C}, \mathcal{D}, W, Z) = (\mathcal{C}, \mathcal{C}, \mathcal{C}, \mathcal{C})$, and that ψ is the identity inclusion, or (b) to insist, in addition to the conditions in the third sentence of the last paragraph, that if $\langle \cdot | \cdot \rangle_{\mathcal{C}}$ is the $\mathcal{C}$-valued inner product on W, then $\langle \psi(f) | \psi(f) \rangle_{\mathcal{C}} = f^2$. We point out that (b) is a weaker hypothesis than (a), i.e., (a) implies (b). Thus we will take our definition of "Af is a strong subequivalence bimodule" to mean (b). We now proceed to show the equivalence of (b) with 6.1 (ii). The fact that (a) is equivalent to 6.1 (ii) will be a step on our route.

In fact, if one looks at the three parts of Definition 5.2 in [7], it will be seen that part (iii) there consists of two (in fact four since we are in the two-sided case) requirements which in some sense are symmetric. The first of these requirements is that $W \cong \mathcal{C} \otimes_{hA} X$ via the product map $\mathcal{C} \times X \to W$, whereas the other requirements are similar; e.g., that $Z \cong Y \otimes_{hA} \mathcal{C}$ via the product map. It is convenient to consider, and characterize, these requirements separately. Or, since they are in some sense symmetric, we will essentially need to only consider the first requirement just mentioned. Thus we shall say that Af satisfies condition (a$'$) (resp. (b$'$)), if it satisfies condition (a) (resp. (b)) with the exception of replacing (iii) of [7] 5.2 by the single requirement that $W \cong \mathcal{C} \otimes_{hA} X$ via the product map. What we will show is that Af satisfies (b$'$) if and only if $f^{-2} \in \mathcal{G}$, where the latter set is defined to be the norm (or uniform) closure in $\mathcal{C}$ of the set of finite sums of terms of the form $|a|^2 = a^*a$, for $a \in A$. In the case that A is a function algebra we have $\mathcal{G} = \mathcal{G}_A(\Omega)$ in the earlier notation of this paper (see §2).

The following result further clarifies hypotheses (a) and (a$'$) above.

Lemma 6.6. *Suppose that A is an operator algebra with an identity of norm 1, which generates a C^*-algebra $\mathcal{C}$. Suppose that f is an invertible element of $\mathcal{C}$. Then (A, B, Af, Y) is a subcontext of $(\mathcal{C}, \mathcal{C}, \mathcal{C}, \mathcal{C})$ (in the sense of* [7]*, Definition 5.2) if and only if $Y = f^{-1}A$ and $B = f^{-1}Af$.*

Proof. We only need to prove one direction, the other being clear from the definition of "subcontext". First we observe that since A and B both generate $\mathcal{C}$, it follows from [6], Lemma 8.1, that $\mathcal{C}$ and also $\mathcal{B}$ are unital, and all have a common identity element. We have from the remark in the third last paragraph of p. 272 in [7] that $AfY = A$ and $YAf = B$. This implies that $fY \subset A$, so that $Y \subset f^{-1}A$. Hence $B = YAf \subset f^{-1}Af$. Writing 1_B as a finite sum $\sum_k y_k a_k f$, we have that $f^{-1}A = \sum_k y_k a_k A \subset Y$. Thus $Y = f^{-1}A$, and $f^{-1}Af = YX = B$. ☐

Certainly the next result in the case that A is a function algebra readily leads to the earlier stated equivalence between definition (a) mentioned above and 6.1 (ii):

Proposition 6.7. *Suppose that A is an operator algebra with an identity of norm 1, which generates a C^*-algebra $\mathcal{C}$. Suppose that f is an invertible element of $\mathcal{C}$. Then the following are equivalent:*

(i) *$\mathcal{C} \otimes_{hA} Af \cong \mathcal{C}$ isometrically via the multiplication map $m : \mathcal{C} \times Af \to \mathcal{C}$;*
(ii) *given any $\epsilon > 0$, there exists a sequence $x_1, \cdots, x_n \in Af$ such that*

$$\frac{1}{1+\epsilon} \leq \sum_{k=1}^{n} x_k^* x_k \leq 1 + \epsilon .$$

Moreover, in this case the isometric isomorphism in (i) *is completely isometric.*

Proof. (ii) $\Rightarrow$ (i): Given (ii), we let $r = \sum_{i=1}^n |x_i|^2$ (interpreting $|x|^2 = x^*x$ if we are in the noncommutative case). To see that $\mathcal{C} \otimes_{hA} Af \cong \mathcal{C}$ (completely) isometrically, one may appeal to [11], Theorem 2.8. Alternatively, here is a direct argument: The product map $m : \mathcal{C} \otimes_{hA} Af \to \mathcal{C}$ is a complete contraction. Define a map $\rho_\epsilon(g) = \sum_{i=1}^n g\, r^{-1} x_i^* \otimes x_i$ from $\mathcal{C} \to \mathcal{C} \otimes_{hA} Af$. Clearly $m \circ \rho_\epsilon = Id_{\mathcal{C}}$. A simple algebraic calculation shows that $\rho_\epsilon(m(g \otimes af)) = g \otimes af$ for all $g \in \mathcal{C}$ and $a \in A$. Thus $\rho_\epsilon \circ m = Id$ too. So $\rho_\epsilon = m^{-1}$. Since $\|\rho_\epsilon\|_{cb} < (1+\epsilon)^2$, it follows that m is a complete isometry, and hence $\mathcal{C} \otimes_{hA} Af \cong \mathcal{C}$.

(i) $\Rightarrow$ (ii): Given (i), by [11], Theorem 2.8, there exist $g_1, \cdots, g_n \in \mathcal{C}$ and $x_1, \cdots, x_n \in X$ such that $1 = \sum_k g_k x_k$, and $\| \sum_k g_k g_k^* \| \leq 1+\epsilon$, and $\| \sum_k x_k^* x_k \| \leq 1+\epsilon$. A well-known principle says that for a nonnegative element a in a unital C^*-algebra, $\|a\| \leq 1$ if and only if $a \leq 1$. A well-known Cauchy-Schwarz argument shows that if g_i and x_i are elements in a unital C^*-algebra, and if $1 = \sum_k g_k x_k$, then $1 \leq \| \sum_k g_k g_k^* \| \, (\sum_k x_k^* x_k)$. Putting these facts together gives the desired equation in (ii). ◻

The following result shows that condition (b′) essentially reduces to (a′).

Theorem 6.8. *Suppose that A is an operator algebra with an identity of norm 1, which generates a C^*-algebra $\mathcal{C}$. Suppose that f is a strictly positive element of $\mathcal{C}$. Suppose that Af satisfies definition* (b′) *above. Then the two equivalent statements of Proposition 6.7 are valid for this f. In particular, $f^{-2} \in \mathcal{G}$.*

If further, condition (b) *holds, then also f^2 is uniformly approximable by finite sums of terms of the form aa^* for $a \in A$. Thus if A is a function algebra on Ω, then f^2 and f^{-2} are in $\mathcal{G}_A(\Omega)$.*

Proof. The product map $m : \mathcal{C} \otimes_{hA} X \to \mathcal{C}$ is a contractive $\mathcal{C}$-module map onto $\mathcal{C}$. The map $\rho : \mathcal{C} \to \mathcal{C} \otimes_{hA} Af$ taking $g \mapsto gf^{-1} \otimes f$ is a bounded $\mathcal{C}$-module map, which, by an argument similar to the proof of 6.7, coincides with m^{-1}. So $W \cong \mathcal{C} \otimes_{hA} Af \cong \mathcal{C}$ as left $\mathcal{C}$-modules, and therefore unitarily as left C^*-modules over $\mathcal{C}$ by a well-known fact about isomorphisms of finitely generated C^*-modules. Let $\varphi : W \to \mathcal{C}$ be the associated isometric surjective left $\mathcal{C}$-module map. By the theory of Morita equivalence of C^*-algebras, we then obtain $\mathcal{D} \cong {}_{\mathcal{C}}B(W, W) \cong \mathcal{C}$. So $\mathcal{D}$ is *-isomorphic to $\mathcal{C}$, and a similar argument shows that $Z \cong \mathcal{C}$ unitarily as right C^*-modules. Since $\mathcal{D}$ is unital, so is B. Indeed there is a 1-1 *-isomorphism π from the "linking C^*-algebra" for W, onto $M_2(\mathcal{C})$, such that the "1-2-corner" of π is φ, and the "1-1-corner" of π is $Id_{\mathcal{C}}$. Let $X' = \varphi(X)$ and let $k = \varphi(h)$, where $h = \psi(f)$ and ψ is as in the discussion a few paragraphs above 6.6. Thus $X' = Ak$. Letting Y' and B' be the subspaces of $\mathcal{C}$ associated with Y and B under π, we then have that (A, X', Y', B') is a subcontext of $(\mathcal{C}, \mathcal{C}, \mathcal{C}, \mathcal{C})$. Moreover, $\mathcal{C} \otimes_{hA} X' \cong \mathcal{C}$ completely isometrically via the product map; to see this, consider the following sequence of complete isometries:

$$\mathcal{C} \otimes_{hA} X' \overset{Id \otimes \varphi^{-1}}{\longrightarrow} \mathcal{C} \otimes_{hA} X \;\to\; W \overset{\varphi}{\to} \mathcal{C}\,.$$

The middle map here is the "product map". The composition of these complete isometries coincides with the product map $\mathcal{C} \otimes_{hA} X' \to \mathcal{C}$.

From the last fact and [11], 2.8, it follows, as in the end of the proof of 6.7, that given $\epsilon > 0$, there exist $g_i \in \mathcal{C}$ and $a_i \in A$, with $1_{\mathcal{C}} = (\sum_i g_i a_i)k$, and with

$\sum_i g_i g_i^* \leq 1 + \epsilon$ and

$$\frac{1}{1+\epsilon} \leq \sum_i k^* a_i^* a_i k \leq 1 + \epsilon \ .$$

From the above, we see that k has a left inverse in $\mathcal{C}$. On the other hand, $kk^* = \varphi(h)\varphi(h)^* = \langle f|f\rangle_{\mathcal{C}}$, since φ is unitary. By hypothesis (b′), we see that $kk^* = f^2$. By assumption this is invertible, so that k is right invertible. Thus k is invertible. Pre- and post-multiplying in the displayed equation in the last paragraph by k and k^* respectively, and then pre- and post-multiplying again by $(kk^*)^{-\frac{1}{2}} = f^{-1}$, shows that (ii) of 6.7 holds.

Now suppose further that (b) holds, or more generally that $Z \cong Y \otimes_{hA} \mathcal{C}$ via the product map. By an argument similar to that at the end of the first paragraph of this proof it follows that $k^{-1}A \ \otimes_{hA} \ \mathcal{C} \cong \mathcal{C}$ $\mathcal{C}$-isometrically. By [11], 2.8, we may then proceed as in the second paragraph, using the fact that $kk^* = f^2$, to obtain the last conclusion. □

For those interested in the "noncommutative versions" of our results, it is interesting to note that the formulation in 6.1 (i) of Af being a strong Morita equivalence bimodule, and its alternative formulation in terms of the class $\mathcal{M}_A$ defined in the introduction, has an almost identical noncommutative version. This no doubt is useful in finding noncommutative examples of strong Morita equivalences of nonselfadjoint operator algebras. Briefly, the result is as follows, the ideas being much the same as in 6.7. If $A, \mathcal{C}, f$ are as in the first line of 6.7, with f strictly positive, then $(A, f^{-1}Af, Af, f^{-1}A)$ is a (P)-context in the sense of [11], 3.1 (which loosely speaking says that it is "half a Morita equivalence", and almost the same as saying that Af is a "rigged module" in the sense of [4]), if and only if $f \in \mathcal{M}_A^l$, where the last space is defined very similarly to $\mathcal{M}_A$. Namely, $f \in \mathcal{M}_A^l$ if and only if there exist A-tuples $K_m, H_m \in A^{n_m}$, with $K_m.H_m = 1$ and $K_m.K_m^* \to f^2$ and $H_m^*.H_m \to f^{-2}$. Also, $(A, f^{-1}Af, Af, f^{-1}A)$ is a Morita context in the sense of [11], 3.1, if and only if $f \in \mathcal{M}_A^l$ and $f \in \mathcal{M}_B^r$, where $B = f^{-1}Af$, and $\mathcal{M}_B^r$ is defined analogously to $\mathcal{M}^l$ above except that $K_m.K_m^*$ is replaced by $K_m^*.K_m$, and similarly for $H_m.H_m^*$. We leave these assertions, which as we said use similar ideas to the proof of 6.7, to the interested reader.

7. The Picard group

Some material in this section, and indeed in the rest of the paper, requires some technical knowledge of several papers. The reader who desires further background is directed to [6] for a leisurely introduction to our work, and to [11, 10, 4] for more specific details. The general reader is advised to simply read the main results below.

We begin by discussing strong Morita equivalence of function algebras. As we said earlier, we shall not give the general definition, but simply say that it involves a pair of bimodules X and Y, called *equivalence bimodules*. A well-known property of (strong) Morita equivalence is that if A and B are unital, commutative and (strongly) Morita equivalent, then A is (isometrically) isomorphic to B. So we shall assume that $A = B$. However, this is not the end of the story, for the question remains as to which A-A-bimodules implement such a "self-equivalence" of A. The collection of such bimodules, with two such bimodules identified if they are (completely) isometrically A-A-isomorphic, is a group, with multiplication being the Haagerup tensor product $\otimes_{hA}$ (see [11]). We call this the *strong Picard group*, and write $Pic_s(A)$. For background on the Picard group, see any algebra text covering Morita equivalence (for example [16], Chapter 12), or [12, 14] for a discussion of the C^*-algebraic version. For example, we will show below that for $A = A(\mathbb{D})$, the disk algebra, this Picard group is the direct product of the Möbius group and the abelian group $C(\mathbb{T})/\operatorname{Re} A$.

In fact, for the most part, we will only consider a certain subgroup of $Pic_s(A)$, namely the singly generated bimodules. We will show that for a function algebra A on a compact (Hausdorff) space Ω, every such bimodule is essentially of the form Af, where f is a strictly positive continuous function on Ω. Indeed, f may be chosen in $\mathcal{M}(\Omega)$, and by 7.6 it follows that f is unique up to the coset Q.

Indeed, even in the non-singly generated case, if the second Čech cohomology group of Ω vanishes, we shall see that every strong Morita equivalence A-A-bimodule is a finitely generated submodule of $C(\Omega)$. Of course, the non-singly generated case is probably much more interesting, but will no doubt require a much deeper analysis. Our intention here is mainly to point out very clearly the features of the singly generated case.

As we said earlier, the multiplication on the Picard group of A, is given by the module Haagerup tensor product $\otimes_{hA}$. See [11] for details about this tensor product. However, in the case that the modules are singly generated (as just discussed), this tensor product becomes rather trivial:

Lemma 7.1. *Suppose that $f \in \mathcal{M}(\Omega)$, and $g \in C(\Omega)^+$. Then $Af \otimes_{hA} Ag \cong A(fg)$ (completely) A-A-isometrically.*

We omit the proof of this, which is a special case of the more general Lemma 8.16, which we prove later.

A standard type of strong Morita equivalence bimodule for A comes from taking an isometric automorphism $\theta : A \to A$, and defining the module $A_\theta = A$, with the usual left module action, and with right module action $b \cdot a = b\theta(a)$. It is easy to check that this is a strong Morita equivalence bimodule for A. The collection of (equivalence classes of) this type of equivalence bimodule is a subgroup of $Pic_s(A)$, which is isomorphic to the group $Aut(A)$ of isometric automorphisms of A. This, in the case that A is a function algebra, corresponds to a group of homeomorphisms of the maximal ideal space of A, which restrict to homeomorphisms of the Shilov boundary of A.

More generally, if X is a strong Morita equivalence A-A-bimodule, and if $\theta \in Aut(A)$, then X_θ is also a a strong Morita equivalence A-A-bimodule. Here X_θ is X but with right action changed to $x \cdot a = x\theta(a)$. One way to see this is to note that $X_\theta \cong X \otimes_{hA} A_\theta$.

An A-A-bimodule X will be called "symmetric" if $ax = xa$ for all $a \in A, x \in X$.

Proposition 7.2. *For a function algebra A, $Pic_s(A)$ is a semidirect product of $Aut(A)$ and the subgroup of $Pic_s(A)$ consisting of symmetric equivalence bimodules. Thus, every $V \in Pic_s(A)$ equals X_θ, for a symmetric $X \in Pic_s(A)$, and for some $\theta \in Aut(A)$.*

Proof. This follows essentially as in pure algebra ([16], Chapter 12.18). Suppose that X is any strong Morita equivalence A-A-bimodule. Then by the basic Morita theory, any right A-module map $T : X \to X$ is simply left multiplication by a fixed element of A. Indeed, via this identification, we have $A \cong CB_A(X)$ isometrically and as algebras ([11], 4.1 and 4.2). For fixed $a \in A$, the operator $x \mapsto xa$ on X is a right A-module map, with completely bounded norm $= \|a\|$. Therefore by the above fact, there is a unique $a' \in A$ such that $a'x = xa$ for all $x \in X$. The map $a \mapsto a''$ is then seen to be an isometric unital automorphism θ of A. In this way we have defined a surjective group homomorphism $Pic_s(A) \to Aut(A)$. This homomorphism has a 1-sided inverse $Aut(A) \to Pic_s(A)$, namely $\theta \to A_\theta$. In this way, we see the "semidirect product" statement. Note that $X = (X_{\theta^{-1}})_\theta$, and $X_{\theta^{-1}}$ is symmetric. ☐

More generally, if X is a strong subequivalence A-A-bimodule in the sense of [7], then a similar argument works. Since we will not use this result here, we will not give full details, but the idea goes as follows: Suppose that X corresponds (using the notation of [7], §5) to a subcontext (A, A, X, Y) of a C^*-Morita context $(\mathcal{C}, \mathcal{D}, W, Z)$, where $\mathcal{C} = C(\Omega)$. It follows that $\mathcal{D} \cong \mathcal{C}$ isometrically. Now use the argument of the proof above, combined with Theorem 5.6 in [7].

Thus we may assume henceforth that X is symmetric. Then X dilates to a strong Morita equivalence $\mathcal{C}$-$\mathcal{C}$-bimodule $W = \mathcal{C} \otimes_{hA} X$. From [4], Theorem 6.8, we know that W contains X completely isometrically. It is helpful to consider the inclusion

$$\begin{bmatrix} A & X \\ Y & A \end{bmatrix} \subset \begin{bmatrix} \mathcal{C} & W \\ Z & \mathcal{C} \end{bmatrix}$$

of linking algebras. Note that since $W \cong \mathcal{C} \otimes_A X$, we have $wa = aw$ for all $w \in W, a \in A$. Similarly, for $Z \cong Y \otimes_A \mathcal{C}$ we have $za = az$ (note that Y is symmetric too—this follows from the fact that X is symmetric, and $Y \cong {}_A HOM(X, A)$ [16]). Since $Z = W^*$, we have $wa^* = (aw^*)^* = (w^*a)^* = a^*w$. Therefore $xw = wx$ for all $w \in W, x \in \mathcal{C}$. Thus W is a symmetric element of $Pic(C(\Omega))$, and consequently (see the appendix of [29], or [31, 12, 14]) W may be characterized as the space of sections of a complex line bundle over Ω.

Proposition 7.3. *Suppose that A is a function algebra on Ω, and that X is a symmetric subequivalence A-A-bimodule (or strong Morita equivalence bimodule). If every complex line bundle over Ω is trivial, or equivalently, if the Čech cohomology group $H^2(\Omega, \mathbb{Z}) = 0$, then $W \cong C(\Omega)$ completely A-A-isometrically. Thus X is A-A-isometric to a finitely generated A-A-submodule of $C(\Omega)$.*

The hypothesis $H^2(\Omega, \mathbb{Z}) = 0$ applies, for example, if $\Omega = \bar{\mathbb{D}}$ or $\mathbb{T}$, or more generally, any multiply connected region in the plane.

Theorem 7.4. *Suppose that A is a function algebra on Ω. Every singly generated strong Morita equivalence (resp. subequivalence) A-A-bimodule is completely A-A-isometrically isomorphic to $(Af)_\theta$, for some $\theta \in Aut(A), f \in C(\Omega)^+$. The converse of this statement is also true, if A is logmodular or if it is a logMorita algebra (resp. convexly approximating in modulus).*

Proof. We use the notation and facts established in the second paragraph below the proof of Proposition 7.2. If X is singly generated over A, then so is W over $\mathcal{C}$, and again it follows that $W \cong \mathcal{C}$ completely A-A-isometrically. Here is one way to see this (which adapts to the noncommutative case): if x is the generator, then x^*x is a strictly positive element in $\mathcal{C}$ (see for example [11], Theorem 7.13). Thus x^*x is invertible, so that $f = |x|$ is invertible in $\mathcal{C}$. Thus $W = \mathcal{C}x \cong \mathcal{C}f = \mathcal{C}$, and $X = Ax \cong Af$.

The "converse" assertion follows from the earlier correspondences between the classes $\bar{Q}^+$, $\mathcal{M}$ and $\mathcal{G}$, and strong Morita equivalence and subequivalence bimodules. $\square$

It can be shown that for X a singly generated strong Morita equivalence A-A-bimodule, the f in the last theorem may be chosen in $\mathcal{M}_A(\Omega)$, and (modulo the θ in 7.2) the strong Morita context (in the sense of [11]) associated with X may be identified with (A, A, Af, Af^{-1}). The latter is a subcontext (in the sense of [7]) of $(\mathcal{C}, \mathcal{C}, \mathcal{C}, \mathcal{C})$. These facts may probably be shown directly from what we have done in the proof, and [4], Theorem 5.10, say. However, another proof of these facts is given in Corollary 9.4 in the last section.

We turn now to "rank-one" modules. We illustrate this concept first at the level of pure algebra. Suppose that A is a commutative unital algebra, and that X is a (purely algebraic) Morita equivalence A-A-bimodule, with "inverse bimodule" Y (see [16]). As above, we say that X is symmetric if $ax = xa$ for all $x \in X, a \in A$. We shall say that X is *algebraically rank 1*, if $1 = (x', y') = [y'', x'']$ for some $x', x'' \in X, y', y'' \in Y$; here $(\cdot)$ and $[\cdot]$ are the Morita pairings [16]. It is easy to show that this implies that $(x, y) = [y, x]$ for all $x \in X, y \in Y$. Simple algebra shows that a symmetric bimodule X is an algebraically rank 1 Morita equivalence A-A-bimodule if and only if $X \cong A$ as A-modules.

Now suppose that X is a symmetric strong Morita equivalence A-A-bimodule. We say that X is *rank one*, if X is algebraically rank 1, and for any $\epsilon > 0$, the x', y' above may be chosen with norms within ϵ of 1. It follows that, if we define $T(x) = (x, y')$ on X, and $S(a) = ax'$ on A, then $S = T^{-1}$. Also the norms $\|T\|_{cb} = \|T\|$ and $\|S\|_{cb} = \|S\|$ are close to 1. Thus it follows that $X \cong A$ almost completely A-isometrically, and we see that X is a MIN space (that is, its operator space structure is that of a subspace of a commutative C^*-algebra).

We can therefore add to our earlier Corollary 4.6, the following:

Corollary 7.5. *Let X be a Banach A-module. The following are equivalent:*

(i) *$X \cong A$ almost A-isometrically;*
(ii) *X, with the symmetric bimodule action, is a rank-one strong Morita equivalence A-A-bimodule;*
(iii) *there exists $f \in \bar{Q}^+$ such that $X \cong Af$ A-isometrically.*

Proof. We just saw that (ii) $\Rightarrow$ (i). As we saw in 4.6, condition (i) is equivalent to (iii). Proposition 6.2 (c) shows that (iii) implies (ii). $\square$

Thus the new definition of rank-one is equivalent to our earlier definition (6.1 (i)) of Af being a rank-one strong Morita equivalence bimodule.

Theorem 7.6. (i) *If $f_1, f_2 \in \mathcal{M}$ then $Af_1 \cong Af_2$ A-isometrically if and only if $f_1 = hf_2$ for some $h \in Q$.*

(ii) *The map $[f] \mapsto Af$ gives an injective group homomorphism $\mathcal{M}' = \mathcal{M}/Q \to Pic_s(A)$. The range of this homomorphism consists of all the topologically singly generated symmetric elements of $Pic_s(A)$. Thus $Pic_s(A)$ contains, as a subgroup, the direct product of $\mathcal{M}'$ and $Aut(A)$.*
(iii) *If one restricts the group homomorphism in* (ii) *to $Q' = \bar{Q}^+/Q$, then its range consists of the rank 1 strong Morita equivalence A-A-bimodules.*
(iv) *If $f_1, f_2 \in \mathcal{M}$, then $Af_1 \cong Af_2$ almost A-isometrically if and only if $f_1 = hf_2$, for some $h \in \bar{Q}^+$.*

Proof. By the "harmonicity" associated with the $\mathcal{M}$-class, we can assume that $f_1, f_2 \in C(\partial A)^+$. Then (i) follows immediately from Corollary 3.7. However, here is another argument, which adapts immediately to give (iv) too. Suppose that $Af_1 \cong Af_2$ A-isometrically, where $f_1, f_2 \in \mathcal{M}$. Then by 7.1 we have $A \cong Af_1^{-1} \otimes_{hA} Af_1 \cong Af_1^{-1} \otimes_{hA} Af_2 \cong A(f_2 f_1^{-1})$. This implies that $f_2 f_1^{-1} \in Q$ by the last theorem. Thus $[f_1] = [f_2]$. This also gives (ii), in view of 7.1. An argument similar to that for (i) also proves (iv). □

For many common function algebras, every strong Morita equivalence A-A-bimodule X is singly generated. We shall illustrate this for $A = A(\mathbb{D})$. Such an X is algebraically finitely generated and projective ([16], 12.7). Then, by Theorem 3.6 in [4], X is completely boundedly A-isomorphic to a closed A-complemented submodule of $A^{(n)}$. The associated projection $P : R_n(A) \to R_n(A)$ may be thought of as an analytic projection valued function $f_P : \mathbb{D} \to M_n$, and since the disk is contractible, f_P is homotopic to the constant function $f_P(0)$. Thus (see [3], 4.3.3) P is similar to a constant projection in M_n. We originally heard this last argument from P. Muhly. Hence $X \cong A^{(m)}$ algebraically, and if $m > 1$, then by [16], $A \cong M_n(A)$ algebraically, which is impossible. So X is singly generated. Putting this together with Theorem 7.6 (ii), and the fact that $A(\mathbb{D})$ is a Dirichlet algebra (so that $\bar{Q}^+ = \mathcal{M} = C(\mathbb{T})^+$), we have:

Corollary 7.7. *The Picard group of $A = A(\mathbb{D})$ is the direct product of the Möbius group and the abelian group $C_{\mathbb{R}}(\mathbb{T}) / \operatorname{Re} A$.*

Remark. We have proved elsewhere that the strong Picard group of A is isomorphic to the group of category equivalences of ${}_A OMOD$ with itself, where ${}_A OMOD$ is the category of left operator modules over A.

Part C

8. Rigged modules over function algebras

A (left) A-Hilbertian module is a left operator module X over A, such that there exist a net of natural numbers n_α, and completely contractive A-module maps $\phi_\alpha : X \to R_{n_\alpha}(A)$ and $\psi_\alpha : R_{n_\alpha}(A) \to X$, such that $\psi_\alpha \circ \phi_\alpha(x) \to x$, for all $x \in X$. Here $R_n(A)$ is $A^{(n)}$, viewed as an operator space by considering it as the first row of $M_n(A)$. The name "Hilbertian" is due to V. Paulsen. An A-rigged module is an A-Hilbertian module with the additional property that for all β we have $\phi_\beta \psi_m \circ \phi_m \to \phi_\beta$ in the cb-norm. If, in addition, X is singly generated, it follows that we can take the net in the definition of A-Hilbertian to be a sequence n_m, ϕ_m, ψ_m, $m \in \mathbb{N}$. We will write $e_m = \psi_m \circ \phi_m$, a completely contractive module map $X \to X$. We will also say that X is *rank 1 Hilbertian* if we can take $n_m = 1$ for all $m \in \mathbb{N}$.

Our main purpose in this section is to show that a singly generated operator module is A-Hilbertian if and only if it is A-rigged, and to attempt to thoroughly understand such modules. It is clear, from the definitions, that every strong Morita equivalence A-A-bimodule is a left A-rigged module, and every left A-rigged module is an A-Hilbertian module. For A a C^*-algebra, we proved in [5, 11] that the converse is true.

If E is a closed subset of Ω, we will write J_E for the ideal $\{f \in A : f(x) = 0 \text{ for all } x \in E\}$ of A. The following result is important for us:

Theorem 8.1 ([33, 21, 15])**.** *Let A be a function algebra on a compact space Ω. If J is an ideal in a function algebra A, then the following are equivalent:*

(i) *J has a contractive approximate identity;*
(ii) *J has a bounded approximate identity;*
(iii) *J is an M-ideal of A;*
(iv) *$J = J_E$, for a p-set E for A in Ω.*

We refer the reader to [17] for details on p-sets and peak sets. We allow $\emptyset$ as a p-set. A p-set is an intersection of peak sets. We will not define the term "M-ideal" here, and it will not play a role.

In (iv), it is clear that the E is unique. Also, $F = E \cap \partial A$ is a p-set for A on the Shilov boundary, and $J_F = J_E$.

It is easy then to see from the definition of a rigged module above that we have:

Corollary 8.2. *If E is a p-set for a function algebra A, then the ideal J_E is an A-rigged module.*

Proof. Let e_α be a c.a.i. for J_E, and define $\psi_\alpha = \phi_\alpha$ to be multiplication by e_α. These satisfy the requirements for a rigged module. □

The proof of the following result requires some knowledge of rigged modules and C^*-modules ([4, 5] and [27, 3]).

Lemma 8.3. *Suppose that $A \subset C(\Omega)$ is a function algebra on a compact space Ω, and that X is a singly generated left A-Hilbertian module. Then there is a nonnegative continuous function f on Ω such that $X \cong (Af)^{\bar{}}$ A-isometrically.*

Proof. As observed in [5], §7, the fact that X is left A-Hilbertian implies that $W = C(\Omega) \otimes_{hA} X$ is a left C^*-module over $C(\Omega)$, and clearly this module is singly generated. Also X may be regarded as an A-submodule of W (in the obvious way). Let f be the single generator of X and W. Suppose that I is the ideal in $\mathcal{C} = C(\Omega)$ generated by the range of the inner product. Now W is a full C^*-module over I, and so IW is dense in W, since C^*-modules are automatically nondegenerate. Since $\mathcal{C}f$ is dense in W, $I\mathcal{C} = I$, and IW is dense in W, we see that If is dense in W. Hence W is singly generated by f over I. The obvious map $F : I \to If \subset W$ is adjointable and F, F^* have dense range; so by the basic theory of C^*-modules (see, e.g., [27], Prop. 3.8), $W \cong I$, I-isometrically. Hence $W \cong I$, $\mathcal{C}$-isometrically. Thus we may view X as an A-submodule of I, and the generator of X generates I as a $\mathcal{C}$-module or I-module. Let us write this generator as f again. Since I is singly generated by f, $|f|^2$, and consequently $|f|$, is a strictly positive element in I, by [11], Theorem 7.13, for example. Of course $|f|$ is not strictly positive on Ω, unless $I = A$. Clearly X is A-isometric to the closure of the submodule $A|f|$ of I. □

It will be useful to have the following fact.

Lemma 8.4. *Suppose that $f \in C(\Omega)_+$, and that $K \in A^{(n)}$. Then $\|a(w)K(w)\|_2 \leq \|af\|_\Omega$ for all $a \in A$ and $w \in \Omega$, if and only if $\|K(w)\|_2 \leq f(w)$ for all $w \in \partial A$.*

Proof. ($\Leftarrow$): For $w \in \partial A$, we have

$$\|a(w)K(w)\|_2 \leq |a(w)f(w)| \leq \|af\|_\Omega \ .$$

Since aK achieves its maximum modulus on ∂A, we have proved this direction.

($\Rightarrow$): If $n = 1$, then this is well known, following by the usual Choquet boundary point argument (such as we've seen, for example, in 3.6). For general n, fix $z \in \mathbb{C}_1^n$.

Then we have $|a(w)(z.K(w))| \leq \|af\|_\Omega$, for all $w \in \Omega$. Thus, by the $n = 1$ case, $|z.K(w)| \leq f(w)$ for all $w \in \partial A$, which gives what we need. □

Corollary 8.5. *A singly generated operator module X over a function algebra A is A-Hilbertian if and only if X is A-rigged.*

Proof. Suppose that X is A-Hilbertian. We may suppose, by 8.3, that $X = (Af)^{\bar{}}$ in $C(\Omega)$. We will take Ω to be the Shilov boundary of A. We use the notation of the beginning of the section, but assume, as we may, that the norms of ϕ_m, ψ_m are strictly less than 1. The map ψ_n may be written as $[a_1, \cdots, a_k] \mapsto \sum_i a_i x_i$ for some $x_i \in X$, and without loss of generality, we can assume that $x_i = h_i f$, with $h_i \in A$. Thus ψ_n may be associated with an A-tuple H_m, and without loss of generality, $f(w)\|H_m(w)\|_2 \leq 1$ for all $w \in \Omega$. See [4], Prop. 2.5(ii).

Also, ϕ_m is completely determined by its action on f; so we can associate ϕ_m with a unique A-tuple K_m. Now it is easily seen that, without loss of generality, e_m may be regarded as (multiplying by) the element $H_m(w).K_m(w)$ of A. We have $fe_m \to f$ uniformly.

Next we note that by Lemma 8.4, the identity $\|\phi_n(af)\| \leq \|af\|_\Omega$ for all $a \in A$ implies that $\|K_m(w)\|_2 \leq f(w)$ for all $w \in \Omega$. Thus it follows that for $a \in A$, we have

$$\|e_m a\|_\Omega \leq C_m \|af\|_\Omega$$

for a constant C_m that does not depend on a.

Set $\phi'_m = \phi_m e_m, \psi'_m = \psi_m$. These give new factorization nets, but now we have that

$$\|\phi'_m - \phi'_m \psi'_n \phi'_n\|_{cb} = \|\phi_m e_m (1 - e_n^2)\|_{cb} \leq \|e_m(1 - e_n^2)\|_\Omega \leq C_m \|f(1 - e_n^2)\|_\Omega \to 0$$

as $n \to \infty$. This says that X is a rigged module. The converse is trivial. □

Definition 8.6. If E is a closed subset of Ω, we define $\mathcal{R}_E(\Omega)$ to be the set of functions $f \in C(\Omega)$ which vanish exactly on the set E, and for which there exist two sequences H_n and K_n of A-tuples such that

(i) $\|K_n(w)\|_2 \leq f(w) \leq \|H_n(w)\|_2^{-1}$ for all $w \in \Omega$ (interpreting $\frac{1}{0} = \infty$), and
(ii) $e_m(w) = H_m(w).K_m(w) \to 1$ uniformly on compact subsets C of $\Omega \setminus E$.

We will write $\mathcal{R}(\Omega)$ for the combined collection of all the $\mathcal{R}_E(\Omega)$ classes.

If we write $g = \log f$, then $f \in \mathcal{R}_E(\Omega)$ if and only if g is finite precisely on E, and there exist H_n, K_n as above, satisfying (ii) and

$$\text{(i}'\text{)} \quad \log \|K_n(w)\|_2 \leq g(w) \leq -\log \|H_n(w)\|_2 \ .$$

It follows from these that $\log \|K_n(w)\|_2$ and $-\log \|H_n(w)\|_2$ converge uniformly to g, on compact subsets of $\Omega \setminus E$ (see Lemma 8.10 below). Thus we see that g is an upper and lower envelope of "sub- and superharmonic" functions. See Proposition 8.11 for more on this. This definition is therefore somewhat reminiscent of the Perron process of solving the Dirichlet problem. See also [18, 19].

Proposition 8.7. $\mathcal{R}_\emptyset(\Omega) = \mathcal{M}(\Omega)$.

We leave this as an exercise for the reader.

Corollary 8.8. *Let A be a function algebra on Ω. Then:*

(i) *If $\mathcal{R}_E(\Omega) \neq \emptyset$, then E is a peak set for A in Ω.*
(ii) *If $f \in \mathcal{R}_E(\Omega)$, then $(Af)^{\bar{}}$ is a rigged module.*

(iii) *If $\Omega = \partial A$ and $f \in C(\Omega)_+$, then $(Af)\bar{}$ is a rigged module if and only if $f \in \mathcal{R}_E(\Omega)$ for some peak set E.*

Proof. If one studies the proof of Corollary 8.5, one sees that the ideas there yield that if $f \in \mathcal{R}_E(\Omega)$ then $(Af)\bar{}$ is a rigged module. If $E = f^{-1}(0)$, then K_n vanishes on E. By (ii) of Definition 8.6, the functions $e_m(w) = H_m(w).K_m(w)$ form a c.a.i. for $C_0(\Omega \setminus E)$, and also for J_E. We deduce from Theorem 8.1 that E is a p-set. Since f is strictly positive on $\Omega \setminus E$, it follows that E is a G_δ, from which we deduce, using [17], II.12.1, that E is a peak set. This gives (i) and (ii).

If $(Af)\bar{}$ is a rigged module, then the ideas of the proof of Corollary 8.5 show that (i) of Definition 8.6 holds for $w \in \partial A$. Set $E = f^{-1}(0)$; then K_m vanishes on the set $F = E \cap \partial A$. We saw in that corollary that $fe_m \to f$ uniformly on ∂A. By the Stone-Weierstrass theorem $C_0(\partial A \setminus F)f$ is dense in $C_0(\partial A \setminus F)$. Thus the functions $e_m(w) = H_m(w).K_m(w)$, which are in J_F, form a c.a.i. for $C_0(\partial A \setminus F)$. Thus by Urysohn's lemma, the restriction of f to ∂A is in $\mathcal{R}_F$. □

Corollary 8.9. *The singly generated rigged modules over a function algebra A are exactly (up to A-isometric isomorphism) the modules of the form $(Af)\bar{}$ for $f \in \mathcal{R}_E(\Omega)$.*

If a subset $E \subset \Omega$ is a peak set for A in Ω, then clearly $F = E \cap \partial A$ is a peak set for A in ∂A . Conversely, any peak set F for A in ∂A may be written $F = E \cap \partial A$ for a unique peak set E for A in Ω. We will therefore sometimes be sloppy, and write $\mathcal{R}_F(\Omega)$ or $\mathcal{R}_D(\Omega)$ for $\mathcal{R}_E(\Omega)$, where D is the unique peak set for A in M_A with $D \cap \Omega = E$.

Lemma 8.10. *If f, H_n, K_n are as in the definition of $\mathcal{R}_E(\Omega)$, then $\|K_n(w)\|_2 \to f(w)$ uniformly on Ω, and $\|H_n(w)\|_2 \to f(w)^{-1}$ uniformly on compact subsets of $\Omega \setminus E$. Also, $\|af\|_{\partial A} = \|af\|_\Omega$ for all $a \in A$.*

Proof. If C is a compact subset of $\Omega \setminus E$, and $\epsilon > 0$ is given, then

$$(1-\epsilon)f(w) \le f(w)|H_m(w).K_m(w)| \le \|K_m(w)\|_2 \le f(w)$$

uniformly for $w \in C$ and m sufficiently large. From this one easily sees that $\|K_n(w)\|_2 \to f(w)$ uniformly on Ω, since f is bounded away from zero on C. The second statement is similar. Finally, for $w \in \Omega$, we see that

$$|a(w)f(w)| \le \sup_m \|a(w)K_m(w)\|_2 \le \sup_m \sup\{\|a(x)K_m(x)\|_2 : x \in \partial A\} \le \|af\|_{\partial A} \ .$$

□

If H is an A-tuple, then it will be useful to think of $\log \|H(w)\|_2$ as a *subharmonic* function. The logarithm of a function $f \in \mathcal{R}_E$, on the other hand, should be thought of as being *harmonic*, as we mentioned briefly before. The following few results begin to justify these assertions:

Proposition 8.11. *Let A be a uniform algebra on a compact space Ω, and suppose that $f \in \mathcal{R}_E(\Omega)$. Then:*

(i) *$f_{|\partial A} \in \mathcal{R}_{E \cap \partial A}(\partial A)$.*
(ii) *If $g = \log f$, then g achieves its maximum and minimum on ∂A.*
(iii) *If there are a domain $R \subset \mathbb{C}^n$ and an inclusion $R \subset \Omega$ such that all functions in A are analytic functions on R, then $g = \log f$ is harmonic (in the usual sense on R) whenever it is finite (that is, on $R \setminus E$).*

It is possible that in (iii) above, $R \setminus E = R$ automatically, if R is a (connected) domain as in (iii). This is the case if A is the disk algebra (see comments after Example 8.14). We have not checked this in general, though.

Proof. (i) is obvious. (ii): Let K_n, H_n be A-tuples as in Definition 8.6. For any unit vector z in the complex Euclidean space of the appropriate dimension, and any $w \in \Omega$, we have

$$|K_n(w).z| \le \|K_n(\cdot).z\|_{\partial A} \le \|f\|_{\partial A} \ .$$

Thus $\log \|K_n(w)\|_2 \le \log \|f\|_{\partial A}$. Letting $n \to \infty$ gives $g(w) \le \sup_{\partial A} g$. To get the other inequality, we may assume that g is bounded below. Thus $E = \emptyset$. As above, we obtain $\|H_n(w)\|_2 \le \sup_{\partial A} |f|^{-1}$. Hence $-\log \|H_n(w)\|_2 \ge \inf_{\partial A} g$. Now let $n \to \infty$.

(iii): A function is harmonic on a domain in $\mathbb{C}^n$ if and only if it satisfies the Mean Value Principle. Let H_n, K_n, z be as in (ii), and fix $w_0 \in R \setminus E$. Since $K_n(w).z$ is analytic for $w \in R$, we have that $\log |K_n(w).z|$ is subharmonic on R. Thus, for any ball B with center w_0 in R, we have

$$\begin{aligned} \log |K_n(w_0).z| &\le \frac{1}{m(B)} \int_B \log |K_n(w).z| \le \frac{1}{m(B)} \int_B \log \|K_n(w)\|_2 \\ &\le \frac{1}{m(B)} \int_B \log f(w) \ . \end{aligned}$$

Since this is true for all such z, we have $\log \|K_n(w_0)\|_2 \le \frac{1}{m(B)} \int_B \log f(w)$. Taking the limit as $n \to \infty$ gives $\log f(w_0) \le \frac{1}{m(B)} \int_B \log f(w)$. A similar argument using H_n gives the other direction of the Mean Value Property. ☐

We state the following obvious fact, since it will be referred to several times:

Lemma 8.12. *Let A be a uniform algebra on a compact space Ω, and suppose that H and K are A-tuples such that $\|H(w)\|_2 \|K(w)\|_2 \le 1$ for all $w \in \Omega$, or all $w \in \partial A$. Then the same inequality holds for all $w \in M_A$.*

Proof. Let z, y be vectors in complex Euclidean space. Then we have

$$|(z \cdot H(w))(y \cdot K(w))| \le 1$$

for all $w \in \Omega$, and consequently for all $w \in M_A$. ☐

Theorem 8.13. *Suppose that E is a peak set for A, and that $f \in \mathcal{R}_E(\Omega)$. Let D be the unique peak set in M_A with $D \cap \Omega = E$. Then there is a unique function $\tilde{f} \in \mathcal{R}_D(M_A)$ such that $\tilde{f}$ restricted to Ω equals f. Also, $\|af\|_\Omega = \|a\tilde{f}\|_{M_A}$ for all $a \in A$, so that $(Af)^{\bar{}}$ in $C(\Omega)$ is A-isometric to $(A\tilde{f})^{\bar{}}$ in $C(M_A)$.*

Proof. As we saw above, condition (ii) of Definition 8.6 is equivalent to saying that the functions $e_m(w) = H_m(w).K_m(w)$, which are in J_E, form a c.a.i. for $C_0(\Omega \setminus E)$. However, $J_E = J_D$, so that e_m is a c.a.i. for J_D. Hence by [17], II.12.5 and II.12.7, taking the function p there to be a continuous strictly positive function which is 1 on D and $< \epsilon$ on a compact subset C of $M_A \setminus D$, we see that $e_m(w) \to 1$ uniformly on C.

Now by Lemma 8.12, we have $|e_m(w)| \le \|H_m(w)\|_2 \|K_m(w)\|_2 \le 1$ for all $w \in \Omega$. This then implies that $\lim_m \|H_m(w)\|_2 \|K_m(w)\|_2 = 1$ uniformly on any compact subset C of $M_A \setminus D$. Since $\|H_m(w)\|_2$ and $\|K_m(w)\|$ are uniformly bounded above on C, we deduce that they are also uniformly bounded away from zero on C. By Lemma 8.12, we have

$$\|H_m(w)\|_2 \|K_n(w)\|_2 \leq 1 \text{ for all } w \in M_A \; . \tag{18}$$

Thus we have $\|H_m(w)\|_2 \|H_n(w)\|_2^{-1} \leq 1+\epsilon$ uniformly on C, for m, n large enough. Thus the sequence $\|H_n(w)\|_2$, and by symmetry the sequence $\|K_n(w)\|_2$, are uniformly Cauchy on C. Let $\tilde{f}$ be the uniform limit of $\|K_n(w)\|_2$ on C. Varying over all compact C gives a well-defined continuous $\tilde{f}$ on $M_A \setminus D$. Clearly $\tilde{f}$ extends f.

We next show that $\tilde{f} \in C_0(M_A \setminus D)$. Let $\epsilon > 0$ be given, and let $C = \{w \in \partial A : f(w) \geq \epsilon\}$. For $\gamma > 0$ to be determined, choose (by [17], II.12, again) $a \in A$ with $a \equiv 1$ on D, $\|a\| \leq 1+\gamma$, and $|a| < \gamma$ on C. For $x \in \partial A, m \in \mathbb{N}$ and any Euclidean vector z of norm 1, we have $|(z \cdot K_m(x))a(x)| \leq 2\epsilon$, if γ is smaller than a certain constant which depends only on $\|f\|_{\partial A}$ and ϵ. Hence $|(z \cdot K_m(w))a(w)| \leq 2\epsilon$, for all $w \in M_A$. Hence $\|K_m(w)\|_2 |a(w)| \leq 2\epsilon$. Letting $m \to \infty$, we see that $\tilde{f}(w)|a(w)| \leq 2\epsilon$. In particular, for $w \in U = |a|^{-1}((1-\epsilon, \infty))$, we have $\tilde{f}(w) \leq \frac{2\epsilon}{1-\epsilon}$. Thus indeed $\tilde{f} \in C_0(M_A \setminus D)$.

Define $\tilde{f}$ to be zero on D. Clearly $\|K_m(w)\|_2 \to \tilde{f}(w)$ for all $w \in M_A$. For $w \notin D$ we obtain from (18) that $\|H_m(w)\|\tilde{f}(w) \leq 1$, and also $\|K_m(w)\|_2 \leq \tilde{f}(w)$. Thus $\tilde{f} \in \mathcal{R}_E(\mathcal{M}_A)$.

Finally, for the uniqueness, we suppose that f_1 with H_m^1, K_m^1, and f_2 with H_m^2, K_m^2 both fulfill the definition of $\mathcal{R}_E(M_A)$. If $f_1(x) = f_2(x)$ for all $x \in \partial A$, then $\|H_m^1(w)\|_2 \|K_m^2(w)\|_2 \leq 1$ on ∂A, and hence, by Lemma 8.12, on M_A. Hence $f_2(w) \leq f_1(w)$ for any $w \notin D$. This obviously implies what we want, by symmetry. □

In view of the previous result we may simply write $\mathcal{R}_E$ for $\mathcal{R}_E(\Omega)$, if we wish. As remarked earlier, we may switch E for the corresponding peak set in M_A or ∂A.

Example 8.14. The nontrivial p-sets E for the disk algebra $A(\mathbb{D})$ coincide with the peak sets, and they are exactly the closed subsets of $\mathbb{T}$ of Lebesgue measure 0 (see [21], for example). By the well-known version of Beurling's theorem for the disk algebra, J_E is a singly generated $A(\mathbb{D})$-module. Hence, for $E \subset \mathbb{T}$ with $|E| = 0$, we have by Corollary 8.2 that J_E is an example of a singly generated rigged module over $A(\mathbb{D})$. Indeed every closed ideal I of $A(\mathbb{D})$ is isometrically isomorphic to some J_E, and consequently is an A-rigged module. This is because, by Beurling's theorem, $I = J_E g$ for a fixed inner function g and peak set E.

Next we find all singly generated rigged modules over $A(\mathbb{D})$. They all turn out to be "rank-one Hermitian".

Theorem 8.15. *Suppose that $f \in C(\mathbb{T})_+$. The following are equivalent:*

(i) $\log f$ *is integrable on* $\mathbb{T}$*;*
(ii) $f = |\phi|$ *for a function* $\phi \in H^\infty \setminus \{0\}$*;*
(iii) $(A(\mathbb{D})f)^{-}$ *is a rigged module over* $A(\mathbb{D})$*;*
(iv) $f \in \mathcal{R}(\mathbb{T})$.

Moreover, every singly generated rigged module over $A(\mathbb{D})$ is $A(\mathbb{D})$-isometric to one of the form in (iii).

Proof. The equivalence of (i) and (ii) is classical ([26], p. 53).

Suppose that X is a singly generated rigged module over $A(\mathbb{D})$. By Corollary 8.9 and the fact mentioned in the previous paragraph, we have $X \cong (A(\mathbb{D})f)^{\bar{}}$ A-isometrically, where E is a subset of $\mathbb{T}$ of Lebesgue measure 0, and $f \in \mathcal{R}_E(\mathbb{T})$. With the earlier notation, we have $\|K_m(w)\|_2 \leq f(w)$, and $\|K_m(w)\|_2 \to f(w)$, for all $w \in \mathbb{T}$. This implies that there is a nonzero function $K \in A(\mathbb{D})$ such that $|K| \leq f$ on $\mathbb{T}$. Then $\log|K| \leq \log f$ a.e. on $\mathbb{T}$. Since $\log|K|$ is integrable on $\mathbb{T}$, so is $\log f$.

Conversely, let $\phi \in H^\infty$, with $f = |\phi|$ continuous on $\bar{\mathbb{D}}$. Let E be the subset of $\mathbb{T}$ on which ϕ vanishes, which is a a closed subset of measure 0. Let $w = \log|\phi|$; then w is integrable. We choose a function k_1 on $\mathbb{T}$ such that $w - \epsilon \leq k_1 \leq w$, and such that k_1 is continuously differentiable wherever it is finite. Indeed, one may assume that k_1 lies in a thin strip about $w - \frac{\epsilon}{2}$. We define

$$k(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, k_1(\theta)\, d\theta\right).$$

Then $k \in A(\mathbb{D})$, and $|k| \leq |\phi|$ on $\mathbb{T}$.

Next, choose an open subset U of $\mathbb{T}$ containing E, such that $\int_U |k_1| < \epsilon$ and $k_1 < -1$ on U. We may suppose that U is a finite collection of disjoint open intervals, of total combined length $< \epsilon$. Choose $k_2 = k_1 + \epsilon$ outside U. On U we define k_2 so that k_2 lies between 0 and w on U, and so that k_2 is finite and continuously differentiable on all of $\mathbb{T}$. It is not hard to see that this is possible. Then we have $\int_U (k_2 - k_1) \leq \int_U |k_1| \leq \epsilon$. We define $h \in A(\mathbb{D})$ by the formula defining k above, but with k_1 replaced by $-k_2$. Then h is nonvanishing on $\mathbb{T}$, and $|k| \leq |\phi| \leq |h|^{-1}$ on $\mathbb{T}$. Setting $r = k_2 - k_1$, we may write, for fixed $z \in \mathbb{D}$,

$$\int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, r(e^{i\theta}) \;= \epsilon \int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z} \;-\; \epsilon \int_U \frac{e^{i\theta}+z}{e^{i\theta}-z} \;+\; \int_U \frac{e^{i\theta}+z}{e^{i\theta}-z}\, r(e^{i\theta}) \;\; .$$

The first of the three terms on the right equals $2\pi\epsilon$. Supposing that $d(z, U) \geq \delta$, we have

$$\left|\frac{e^{i\theta}+z}{e^{i\theta}-z}\right| \leq \frac{2}{\delta} \;\; ,$$

for $e^{i\theta} \in U$, whence

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, (k_2 - k_1)(e^{i\theta})\, d\theta\right| \;\leq\; \epsilon(1 + \frac{4}{\delta})\;.$$

We now check that (ii) of Definition 8.6 holds. By an easy compactness argument, we may assume that the compact subset $C \subset \Omega \setminus E$ there is a finite closed interval. Pick ϵ so small in relation to $d(C, E)$ that for any z close enough to C, we have $d(z, s) \geq \sqrt{\epsilon}$ for any $s \in \mathbb{T}$ with $d(s, E) < \epsilon$. For the k_1, k_2 associated with this ϵ, we have, for z close to C,

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}(k_2 - k_1)(e^{i\theta})\, d\theta\right| \leq 5\sqrt{\epsilon}.$$

Thus

$$\left|\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}(k_1 - k_2)(e^{i\theta})\, d\theta\right) \;-\; 1\right| \leq 6\sqrt{\epsilon}\;\; ,$$

for all z close enough to C. Thus $|k(e^{i\theta})h(e^{i\theta}) - 1| \leq 6\sqrt{\epsilon}$, on C. From this it is clear that the conditions of Definition 8.6 are met, so that Af is a rigged module over $A(\mathbb{D})$. □

One may always choose the ϕ in the last theorem to be an outer function, and then $|\phi|$ will also be the "unique harmonic extension" of f to $M_A = \bar{\mathbb{D}}$.

In this case of $A = A(\mathbb{D})$, we see that any $f \in \mathcal{R}$ is nonvanishing inside $\mathbb{D}$, and has a harmonic logarithm on all of $\mathbb{D}$.

Proposition 8.16. *The set $\mathcal{R}(\Omega)$ is a unital semigroup. Indeed, if $f_1 \in \mathcal{R}(\Omega)$ and $f_2 \in C(\Omega)_+$ then $(Af_1)\bar{} \otimes_{hA} (Af_2)\bar{} \cong (Af_1 f_2)\bar{}$ (completely) A-isometrically. If $f_1 \in \mathcal{R}_{E_1}$ and $f_2 \in \mathcal{R}_{E_2}$, then $f_1 f_2 \in \mathcal{R}_{E_1 \cup E_2}$.*

Proof. Clearly the multiplication map $\Phi : (Af_1)\bar{}\otimes_{hA}(Af_2)\bar{} \to (Af_1 f_2)\bar{}$ is completely contractive, and has dense range. Conversely, choose H_m, K_m as in Definition 8.6, and let $e_m = H_m \cdot K_m$ as before. For $a \in A$, define $\theta_m(af_1 f_2) = e_m f_1 \otimes f_2 a = H_m f_1 \odot K_m f_2 a$. The $\odot$ notation here is commonly used with reference to the Haagerup tensor product. Namely, for two finite tuples $x = (x_i), y = (y_i)$, the expression $x \odot y$ means $\sum_i x_i \otimes y_i$. If $g = \chi_E f_1^{-1}$, then $\theta_m(af_1 f_2) = H_m f_1 \odot K_m g(af_1 f_2)$. From the definition of the Haagerup tensor product [11] we see from this that θ_m is well defined and completely contractive. It is easy to see that $\theta_m(\Phi(u)) \to u$, for all $u \in Af_1 \otimes_A Af_2$. Hence Φ is a complete isometry.

We leave it to the reader to check that if $f_1, f_2 \in \mathcal{R}(\Omega)$, then $f_1 f_2 \in \mathcal{R}(\Omega)$. Thus $\mathcal{R}(\Omega)$ is a unital semigroup. The last assertion is also an easy exercise. □

Thus $\log \mathcal{R}_E$ and $\log \mathcal{R}(\Omega)$ are "harmonic classes" in the sense of the introduction.

9. Applications to Morita bimodules

Theorem 9.1. *If X is an algebraically singly generated faithful function module (or equivalently, of the form Ag for some $g \in C(\Omega)^+$), the following are equivalent:*

(i) *$X \cong Af$ A-isometrically, for some $f \in \mathcal{M}$.*

(ii) *For any $\epsilon > 0$, there exist $n \in \mathbb{N}$, and A-module maps $\varphi : X \to A^{(n)}$ and $\psi : A^{(n)} \to X$, with $\psi \circ \varphi = Id_X$, and $\|\varphi\|_{cb} \leq 1+\epsilon$ and $\|\psi\|_{cb} = \|\psi\| \leq 1+\epsilon$.*

(iii) *X is A-rigged.*

In (ii), the operator space structure on $A^{(n)}$ is $R_n(A)$, as in the definition of A-Hilbertian.

Proof. (i) ⇒ (ii): This follows from the definition of $\mathcal{M}$; given $\epsilon > 0$, there exist A-tuples H, K with $H.K = 1$, such that the norms of Hf and Kf^{-1} are close to 1. These may be associated with maps φ and ψ as in the proof of Corollary 8.5.

(ii) ⇒ (iii): This follows from the definition of A-Hilbertian, and the fact that such an X is A-rigged if and only if it is A-Hilbertian (Corollary 8.5).

(iii) ⇒ (ii): The maps ϕ_m, ψ_m in the definition of A-Hilbertian may be associated, as in the proof of Corollary 8.5, with certain A-tuples H_m, K_m. We have $(H_m.K_m)f = \psi_m(\phi_m(f)) \to f$. Hence $(H_m.K_m) \to 1$, so that by the common Neumann series trick, $H_m.K_m$ is an invertible element b^{-1} of A, and $|b| \approx 1$. Replace K_m by $K_m b$. Correspondingly we get an adjusted ϕ'_m, ψ'_m, which now satisfy (ii).

(ii) ⇒ (i) we state as the next result: □

The following result is highly analagous to Theorem 4.3:

Proposition 9.2. *Suppose that A is a uniform algebra on a compact space Ω, and that $f \in C(\Omega)^+$. Then $X = Af$ satisfies condition* (ii) (*or equivalently* (iii)) *of the above theorem if and only if $Af \cong (Af)_{|\partial A}$ isometrically via the restriction map, and $f_{|\partial A} \in \mathcal{M}(\partial A)$.*

Proof. The ($\Leftarrow$) direction is ((i) $\Rightarrow$ (ii)) of the previous theorem.

Supposing (ii) of the theorem, we proceed as in the proof of Corollary 8.5, to associate with φ and ψ, A-tuples H and K. We have that $1 = H.K$, and as in that proof we get $f(\cdot)\|H(\cdot)\|_2 \leq 1+\epsilon$ on Ω, and $\|K(\cdot)\|_2 \leq (1+\epsilon)f(\cdot)$ on ∂A. Applying Cauchy-Schwarz to $1 = |(Hf).(Kf^{-1})|$, and using these inequalities, shows that $(1+\epsilon)\|K(\cdot)\|_2 \geq f$ on Ω, and $\|H(\cdot)\|_2 \approx f^{-1}$ on ∂A. Thus $f_{|\partial A} \in \mathcal{M}(\partial A)$. That $Af \cong (Af)_{|\partial A}$ follows as in the proof of 5.2. ☐

Corollary 9.3. *Suppose that X is a* singly generated *left A-rigged module. Then the following are equivalent:*

(i) *The peak set E associated with X is the empty set.*

(ii) *X, with the obvious (symmetric) right module action, is a strong Morita equivalence A-A-bimodule.*

(iii) *X is algebraically singly generated.*

If M_A is connected, then the above are also equivalent to:

(iv) *X is* algebraically *finitely generated and projective as a left A-module.*

Proof. (i) $\Rightarrow$ (ii): Follows since $\mathcal{M} = \mathcal{R}_\emptyset$.

(ii) $\Rightarrow$ (i), (iii): Assuming (ii), then by Theorem 7.4, $X \cong Af$ A-isometrically, where $f \in C(\partial A)^+$. Since every strong Morita equivalence bimodule is a rigged module, Corollary 8.8 (iii) and the definition of $f \in \mathcal{R}_E$ (see 8.6) now give (i).

(ii) $\Rightarrow$ (iv): This is true for any algebraic Morita equivalence A-A-bimodule [16].

(iv) $\Rightarrow$ (i): It follows by [4], Theorem 3.6 (6), and 9.5 below, that $\mathbb{K}(X) = J_E$ is unital. Thus χ_E is continuous, so that E is closed and open. If Ω is connected, it follows that E is the empty set.

(iii) $\Rightarrow$ (ii): Every singly generated A-rigged module is of the form $(Af)\bar{}$, which is a faithful function A-module. Now appeal to ((iii) $\Rightarrow$ (i)) of Theorem 9.1. ☐

The proof of (ii) $\Rightarrow$ (i) $\Rightarrow$ (ii) above, together with the basic theory of strong Morita equivalence (see [11], section 4), gives a strengthened form of Theorem 7.4:

Corollary 9.4. *Let X be a singly generated symmetric strong Morita equivalence A-A-bimodule. Then there exists an $f \in \mathcal{M}$ such that $X \cong Af$ A-isometrically, $\tilde{X} \cong B_A(X, A) \cong Af^{-1}$ A-isometrically, and the strong Morita context associated with X may be identified with (A, A, Af, Af^{-1}).*

Finally, we show that every singly generated rigged module over a function algebra is a strong Morita equivalence bimodule (over a possibly different algebra):

Theorem 9.5. (i) *If X is a singly generated operator module over a function algebra A on a compact space Ω, then X is an A-rigged module if and only if there exist a peak set E in Ω for A, and a function $f \in \mathcal{R}_E(\Omega)$, such that $X \cong (J_E f)\bar{}$ A-isometrically.*

(ii) *If the equivalent conditions in* (i) *hold, then X, with the obvious (symmetric) right module action, is a strong Morita equivalence J_E-J_E-bimodule.*

(iii) *Conversely, if E is a p-set, then any strong Morita equivalence J_E-J_E-bimodule, or more generally any J_E-rigged module, is an A-rigged module.*

Proof. The proof requires some technical knowledge of rigged modules [4].

Here is one way to see (iii). If X is a left J_E-rigged module, then from Corollary 8.2 and §6 of [4], $J_E \otimes_{hJ_E} X$ is a left A-rigged module. But $J_E \otimes_{hJ_E} X \cong X$, A-isometrically.

To get (i) and (ii), suppose that X is a singly generated A-rigged module. Then, by the previous results, there are a peak set E and a function $f \in \mathcal{R}_E(\Omega)$ such that $X \cong (Af)\bar{}$ A-isometrically. Since f is a strictly positive element of the ideal I in Lemma 8.3, we see that $I = \{p \in C(\Omega) : p(x) = 0 \text{ for all } x \in E\}$. Suppose that $Y = \tilde{X}$ is the dual rigged module of X (see [4, 11] for details, and also for the definition of $\mathbb{K}(X)$, which we shall need shortly). Since W may be taken to be I, it follows that the linking C^*-algebra for W is $M_2(I)$. Thus we can make the following deductions from Theorem 5.10 in [4]. First, $\mathbb{K}(X)$ may be identified with a closed subalgebra J of I, and J has a contractive approximate identity which is a contractive approximate identity for I. Also, Y may be regarded as a subspace of I, and the canonical pairings $X \times Y \to A$ and $Y \times X \to \mathbb{K}(X) = J$ may be regarded as the commutative multiplication in I. Thus it follows that J is the closure of the span of the range of the canonical pairing $X \times Y \to A$. Thus J is also a subset, indeed a closed ideal, of A. By the commutativity of the multiplication in I, X is a strong Morita equivalence J-J-bimodule. We also see that JX is dense in X.

Since J has a contractive approximate identity, we may appeal to Theorem 8.1 to see that $J = J_{E'}$ for a p-set E'. Since J contains a c.a.i. for I, it is clear that $E' \subset E$. On the other hand, if $w \in E \setminus E'$, then there is a peak set F containing E', with $w \notin F$. Choose $a \in A$ with $a \equiv 0$ on E' and $a(w) \neq 0$. Then $a \in J$, and so $a \in I$. This is impossible, so that $E = E'$.

Note that Af is dense in X, JX is dense in X and $JA = J$. Hence Jf is dense in X. Thus $X = (Jf)\bar{}$. □

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